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Abstract. In this paper, we consider inverse scattering and inverse boundary value problems at sufficiently large and fixed energy for the multidimensional relativistic and nonrelativistic Newton equations in a static external electromagnetic field (V, B), $V \in C^2$, $B \in C^1$ in classical mechanics. Developing the approach going back to Gerver-Nadirashvili 1983's work on an inverse problem of mechanics, we obtain, in particular, theorems of uniqueness.

1 Introduction

1.1 *Relativistic Newton equation.* Consider the Newton-Einstein equation in a static electromagnetic field in an open subset Ω of $\mathbb{R}^n$, $n \geq 2$,

$$\begin{aligned} &\dot{p} = -\nabla V(x) + \tfrac{1}{c} B(x)\dot{x}, \\ &p = \frac{\dot{x}}{\sqrt{1 - \frac{|\dot{x}|^2}{c^2}}}, \end{aligned} \tag{1.1}$$

where $x = x(t)$ is a C^1 function with values in Ω, $\dot{p} = \frac{dp}{dt}$, $\dot{x} = \frac{dx}{dt}$, and $V \in C^2(\bar{\Omega}, \mathbb{R})$ (i.e. there exists $\tilde{V} \in C^2(\mathbb{R}^n, \mathbb{R})$ such that $\tilde{V}$ restricted to $\bar{\Omega}$ is equal to V), $B \in \mathcal{F}_{mag}(\bar{\Omega})$ where $\mathcal{F}_{mag}(\bar{\Omega})$ is the family of magnetic fields on $\bar{\Omega}$, i.e. $\mathcal{F}_{mag}(\bar{\Omega}) = \{B' \in C^1(\bar{\Omega}, A_n(\mathbb{R})) \mid B' = (B'_{i,k}),\ \frac{\partial}{\partial x_i} B'_{k,l}(x) + \frac{\partial}{\partial x_l} B'_{i,k}(x) + \frac{\partial}{\partial x_k} B'_{l,i}(x) = 0,\ x \in \bar{D},\ i, k, l = 1 \ldots n\}$ and $A_n(\mathbb{R})$ denotes the space of $n \times n$ real antisymmetric matrices.

By $\|V\|_{C^2,\Omega}$ we denote the supremum of the set $\{|\partial_x^j V(x)| \mid x \in \Omega,\ j = (j_1, .., j_n) \in (\mathbb{N} \cup \{0\})^n, \sum_{i=1}^n j_i \leq 2\}$ and by $\|B\|_{C^1,\Omega}$ we denote the supremum of the set $\{|\partial_x^j B_{i,k}(x)| \mid x \in \Omega,\ \ i, k = 1 \ldots n,\ j = (j_1, .., j_n) \in (\mathbb{N} \cup \{0\})^n, \sum_{i=1}^n j_i \leq 1\}$.

The equation (1.1) is an equation for $x = x(t)$ and is the equation of motion in $\mathbb{R}^n$ of a relativistic particle of mass $m = 1$ and charge $e = 1$ in an external electromagnetic field described by V and B (see [E] and, for example, Section 17 of [LL]). In this equation x is the position of the particle, p is its impulse, t is the time and c is the speed of light.

For the equation (1.1) the energy

$$E = c^2 \sqrt{1 + \frac{|p(t)|^2}{c^2}} + V(x(t)) \tag{1.2}$$

is an integral of motion. We denote by B_c the euclidean open ball whose radius is c and whose centre is 0.

In this paper we consider the equation (1.1) in two situations. We study equation (1.1) when

$$\begin{aligned}&\Omega = D \text{ where } D \text{ is a bounded strictly convex in the strong sense}\\&\text{open domain of } \mathbb{R}^n, n \geq 2, \text{with } C^2 \text{ boundary.}\end{aligned} \tag{1.3}$$

And we study equation (1.1) when

$$\begin{aligned}&\Omega = \mathbb{R}^n \text{ and } |\partial_x^{j_1} V(x)| \leq \beta_{|j_1|}(1+|x|)^{-\alpha-|j_1|},\ x \in \mathbb{R}^n,\ n \geq 2,\\&|\partial_x^{j_2} B_{i,k}(x)| \leq \beta_{|j_2|+1}(1+|x|)^{-\alpha-1-|j_2|},\ x \in \mathbb{R}^n,\end{aligned} \tag{1.4}$$

for $|j_1| \leq 2, |j_2| \leq 1,\ i,k = 1\ldots n$ and some $\alpha > 1$ (here j is the multiindex $j \in (\mathbb{N} \cup \{0\})^n$, $|j| = \sum_{i=1}^n j_i$ and $\beta_{|j|}$ are positive real constants).

For the equation (1.1) under condition (1.3), we consider boundary data. For equation (1.1) under condition (1.4), we consider scattering data.

1.2 *Boundary data.* For the equation (1.1) under condition (1.3), one can prove that at sufficiently large energy E (i.e. $E > E(\|V\|_{C^2,D}, \|B\|_{C^1,D}, D)$), the solutions x at energy E have the following properties (see Properties 2.1 and 2.2 in Section 2 and see Section 6):

$$\begin{aligned}&\text{for each solution } x(t) \text{ there are } t_1, t_2 \in \mathbb{R}, t_1 < t_2, \text{ such that}\\&x \in C^3([t_1,t_2], \mathbb{R}^n), x(t_1), x(t_2) \in \partial D, x(t) \in D \text{ for } t \in]t_1, t_2[,\\&x(s_1) \neq x(s_2) \text{ for } s_1, s_2 \in [t_1, t_2], s_1 \neq s_2;\end{aligned} \tag{1.5}$$

$$\begin{aligned}&\text{for any two distinct points } q_0, q \in \bar{D}, \text{ there is one and only one solution}\\&x(t) = x(t, E, q_0, q) \text{ such that } x(0) = q_0, x(s) = q \text{ for some } s > 0.\end{aligned} \tag{1.6}$$

Let (q_0, q) be two distinct points of ∂D. By $s_{V,B}(E, q_0, q)$ we denote the time at which $x(t, E, q_0, q)$ reaches q from q_0. By $k_{0,V,B}(E, q_0, q)$ we denote the velocity vector $\dot{x}(0, E, q_0, q)$. By $k_{V,B}(E, q_0, q)$ we denote the velocity vector $\dot{x}(s_{V,B}(E, q_0, q), E, q_0, q)$. We consider $k_{0,V,B}(E, q_0, q)$, $k_{V,B}(E, q_0, q)$, $q_0, q \in \partial D, q_0 \neq q$, as the boundary value data.

Remark 1.1. For $q_0, q \in \partial D, q_0 \neq q$, the trajectory of $x(t, E, q_0, q)$ and the trajectory of $x(t, E, q, q_0)$ are distinct, in general.

Note that in the present paper we always assume that the aforementioned real constant $E(\|V\|_{C^2,D}, \|B\|_{C^1,D}, D)$, considered as function of $\|V\|_{C^2,D}$ and $\|B\|_{C^1,D}$, satisfies

$$E(\lambda_1, \lambda_2, D) \leq E(\lambda_1', \lambda_2', D) \text{ if } \lambda_1 \leq \lambda_1' \text{ and } \lambda_2 \leq \lambda_2', \tag{1.7}$$

for $\lambda_1, \lambda_2, \lambda_1', \lambda_2' \in [0, +\infty[$.

1.3 *Scattering data.* For the equation (1.1) under condition (1.4), the following is valid (see [Y]): for any $(v_-, x_-) \in B_c \times \mathbb{R}^n,\ v_- \neq 0$, the equation (1.1) has a unique solution $x \in C^2(\mathbb{R}, \mathbb{R}^n)$ such that

$$x(t) = v_- t + x_- + y_-(t), \tag{1.8}$$

where $\dot{y}_-(t) \to 0,\ y_-(t) \to 0$, as $t \to -\infty$; in addition for almost any $(v_-, x_-) \in B_c \times \mathbb{R}^n,\ v_- \neq 0$,

$$x(t) = v_+ t + x_+ + y_+(t), \tag{1.9}$$

where $v_+ \neq 0$, $|v_+| < c$, $v_+ = a(v_-, x_-)$, $x_+ = b(v_-, x_-)$, $\dot{y}_+(t) \to 0$, $y_+(t) \to 0$, as $t \to +\infty$.

For an energy $E > c^2$, the map $S_E : \mathbb{S}_E \times \mathbb{R}^n \to \mathbb{S}_E \times \mathbb{R}^n$ (where $\mathbb{S}_E = \{v \in B_c \mid |v| = c\sqrt{1 - \left(\frac{c^2}{E}\right)^2}\}$) given by the formulas

$$v_+ = a(v_-, x_-),\ x_+ = b(v_-, x_-), \tag{1.10}$$

is called the scattering map at fixed energy E for the equation (1.1) under condition (1.4). By $\mathcal{D}(S_E)$ we denote the domain of definition of S_E. The data $a(v_-, x_-)$, $b(v_-, x_-)$ for $(v_-, x_-) \in \mathcal{D}(S_E)$ are called the scattering data at fixed energy E for the equation (1.1) under condition (1.4).

1.4 *Inverse scattering and boundary value problems.* In the present paper, we consider the following inverse boundary value problem at fixed energy for the equation (1.1) under condition (1.3):

Problem 1 : given $k_{V,B}(E, q_0, q)$, $k_{0,V,B}(E, q_0, q)$ for all $q_0, q \in \partial D$, $q_0 \neq q$, at fixed sufficiently large energy E, find V and B.

The main results of the present work include the following theorem of uniqueness for Problem 1.

Theorem 1.1. *At fixed* $E > E(\|V\|_{C^2,D}, \|B\|_{C^1,D}, D)$, *the boundary data* $k_{V,B}(E, q_0, q)$, $(q_0, q) \in \partial D \times \partial D$, $q_0 \neq q$, *uniquely determine* V, B.

At fixed $E > E(\|V\|_{C^2,D}, \|B\|_{C^1,D}, D)$, *the boundary data* $k_{0,V,B}(E, q_0, q)$, $(q_0, q) \in \partial D \times \partial D$, $q_0 \neq q$, *uniquely determine* V, B.

Theorem 1.1 follows from Theorem 3.1 given in Section 3.

In the present paper, we also consider the following inverse scattering problem at fixed energy for the equation (1.1) under condition (1.4):

Problem 2 : given S_E at fixed energy E, find V and B.

The main results of the present work include the following theorem of uniqueness for Problem 2.

Theorem 1.2. *Let $\lambda \in \mathbb{R}^+$ and let D be a bounded strictly convex in the strong sense open domain of $\mathbb{R}^n$ with C^2 boundary. Let $V_1, V_2 \in C_0^2(\mathbb{R}^n, \mathbb{R})$, $B_1, B_2 \in C_0^1(\mathbb{R}^n, A_n(\mathbb{R})) \cap \mathcal{F}_{mag}(\mathbb{R}^n)$, $\max(\|V_1\|_{C^2,D}, \|V_2\|_{C^2,D}, \|B_1\|_{C^1,D}, \|B_2\|_{C^1,D}) \le \lambda$, and $\mathrm{supp}(V_1) \cup \mathrm{supp}(V_2) \cup \mathrm{supp}(B_1) \cup \mathrm{supp}(B_2) \subseteq D$. Let S_E^μ be the scattering map at fixed energy E subordinate to (V_μ, B_μ) for $\mu = 1, 2$. Then there exists a nonnegative real constant $E(\lambda, D)$ such that for any $E > E(\lambda, D)$, $(V_1, B_1) \equiv (V_2, B_2)$ if and only if $S_E^1 \equiv S_E^2$.*

Theorem 1.2 follows from Theorem 1.1, (1.7) and Proposition 2.1 of Section 2.

Remark 1.2. Note that for $V \in C_0^2(\mathbb{R}^n, \mathbb{R})$, if $E < c^2 + \sup\{V(x) \mid x \in \mathbb{R}^n\}$ then S_E does not determine uniquely V.

Remark 1.3. Theorems 1.1 and 1.2 give uniqueness results. In this paper we do not prove and do not obtain stability results for Problem 1 and for Problem 2.

1.5 *Historical remarks.* An inverse boundary value problem at fixed energy and at high energies was studied in [GN] for the multidimensional nonrelativistic Newton equation (without magnetic field) in a bounded open strictly convex domain. In [GN] results of uniqueness and stability for the inverse boundary value problem at fixed energy are derived from results for the problem of determining an isotropic Riemannian metric from its hodograph (for this geometrical problem, see [MR], [B] and [BG]).

Novikov [N2] studied inverse scattering for nonrelativistic multidimensional Newton equation (without magnetic field). Novikov [N2] gave, in particular, a connection between the inverse scattering problem at fixed energy and Gerver-Nadirashvili's inverse boundary value problem at fixed energy. Developing the approach of [GN] and [N2], the author [J3] studied an inverse boundary problem and inverse scattering problem for the multidimensional relativistic Newton equation (without magnetic field) at fixed energy. In [J3] results of uniqueness and stability are obtained.

Inverse scattering at high energies for the relativistic multidimensional Newton equation was studied by the author (see [J1], [J2]).

As regards analogs of Theorems 1.1, 1.2 and Proposition 2.1 for the case $B \equiv 0$ for nonrelativistic quantum mechanics see [N1], [NSU], [N3] and further references therein. As regards an analog of Theorem 1.2 for the case $B \equiv 0$ for relativistic quantum mechanics see [I]. As regards analogs of Theorems 1.1, 1.2 for the case $B \not\equiv 0$ for nonrelativistic quantum mechanics, see [ER], [NaSuU] and further references given therein.

As regards results given in the literature on inverse scattering in quantum mechanics at high energy limit see references given in [J2].

1.6 *Structure of the paper.* The paper is organized as follows. In Section 2, we give some properties of boundary data and scattering data and we connect the inverse scattering problem at fixed energy to the inverse boundary value problem at fixed energy. In Section 3, we give, actually, a proof of Theorem 1.1 (based on Theorem 3.1 formulated in Section 3). Section 4 is devoted to the proof of Lemma 3.1 and Theorem 3.1 formulated in Section 3. Section 5 is devoted to the proof of Lemma 2.1 and Proposition 3.1 formulated in Section 2 and in Section 3. Section 6 is devoted to the proof of Properties (2.1) and (2.2). In Section 7, we give results similar to Theorems 1.1, 1.2 for the nonrelativistic case.

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2 Scattering data and boundary value data

2.1 *Properties of the boundary value data.* Let D be a bounded strictly convex open subset of $\mathbb{R}^n$, $n \geq 2$, with C^2 boundary.

At fixed sufficiently large energy E (i.e. $E > E(\|V\|_{C^2,D}, \|B\|_{C^1,D}, D)$ $\geq c^2 + \sup_{x\in\bar{D}} V(x)$) solutions $x(t)$ of the equation (1.1) under condition (1.3) have the following properties (see Section 6):

$$\begin{array}{l} \text{for each solution } x(t) \text{ there are } t_1, t_2 \in \mathbb{R}, t_1 < t_2, \text{ such that} \\ x \in C^3([t_1, t_2], \mathbb{R}^n), x(t_1), x(t_2) \in \partial D, x(t) \in D \text{ for } t \in]t_1, t_2[, \\ x(s_1) \neq x(s_2) \text{ for } s_1, s_2 \in [t_1, t_2], s_1 \neq s_2, \dot{x}(t_1) \circ N(x(t_1)) < 0 \\ \text{and } \dot{x}(t_2) \circ N(x(t_2)) > 0, \text{ where } N(x(t_i)) \text{ is the unit outward} \\ \text{normal vector of } \partial D \text{ at } x(t_i) \text{ for } i = 1, 2; \end{array} \tag{2.1}$$

$$\begin{array}{l} \text{for any two points } q_0, q \in \bar{D}, q \neq q_0, \text{ there is one and only one solution} \\ x(t) = x(t, E, q_0, q) \text{ such that } x(0) = q_0, x(s) = q \text{ for some } s > 0; \\ \dot{x}(0, E, q_0, q) \in C^1((\bar{D} \times \bar{D})\backslash\bar{G}, \mathbb{R}^n), \text{ where } \bar{G} \text{ is the diagonal in } \bar{D} \times \bar{D}; \end{array} \tag{2.2}$$

where $\circ$ denotes the usual scalar product on $\mathbb{R}^n$ (and where by "$\dot{x}(0, E, q_0, q) \in C^1((\bar{D} \times \bar{D})\backslash\bar{G}, \mathbb{R}^n)$" we mean that $\dot{x}(0, E, q_0, q)$ is the restriction to $(\bar{D} \times \bar{D})\backslash\bar{G}$ of a function which belongs to $C^1((\mathbb{R}^n \times \mathbb{R}^n)\backslash\Delta)$ where Δ is the diagonal of $\mathbb{R}^n \times \mathbb{R}^n$).

Remark 2.1. If $B \in C^1(\bar{D}, A_n(\mathbb{R}))$ and $B \notin \mathcal{F}_{mag}(\bar{D})$ (where $A_n(\mathbb{R})$ denotes the space of $n \times n$ real antisymmetric matrices), then at fixed energy $E > E(\|V\|_{C^2,D}, \|B\|_{C^1,D}, D)$ solutions $x(t)$ of equation (1.1) under condition (1.3) also have properties (2.1), (2.2) (see Section 6).

We remind that the aforementioned real constant $E(\|V\|_{C^2,D}, \|B\|_{C^1,D}, D)$, considered as function of $\|V\|_{C^2,D}$ and $\|B\|_{C^1,D}$, satisfies (1.7). In addition, real constant $E(\|V\|_{C^2,D}, \|B\|_{C^1,D}, D)$ has the following property: for any C^2 continuation $\tilde{V}$ of V on $\mathbb{R}^n$, and for any $\tilde{B} \in C^1(\mathbb{R}^n, A_n(\mathbb{R}))$ such that $\tilde{B} \equiv B$ on $\bar{D}$, one has

$$E(\|\tilde{V}\|_{C^2,D_{x_0,\varepsilon}}, \|\tilde{B}\|_{C^1,D_{x_0,\varepsilon}}, D_{x_0,\varepsilon}) \to E(\|V\|_{C^2,D}, \|B\|_{C^1,D}, D), \text{ as } \varepsilon \to 0, \tag{2.3}$$

where $D_{x_0,\varepsilon} = \{x_0 + (1+\varepsilon)(x - x_0) \mid x \in D\}$ for any $x_0 \in D$ and $\varepsilon > 0$ (note that $D_{x_0,\varepsilon}$ is a bounded, open, strictly convex (in the strong sense) domain of $\mathbb{R}^n$ with C^2 boundary).

Let $E > E(\|V\|_{C^2,D}, \|B\|_{C^1,D}, D)$. Consider the solution $x(t, E, q_0, q)$ from (2.2) for $q_0, q \in \bar{D}$, $q_0 \neq q$. We define vectors $k_{V,B}(E, q_0, q)$ and $k_{0,V,B}(E, q_0, q)$ by

$$\begin{aligned} k_{V,B}(E, q_0, q) &= \dot{x}(s_{V,B}(E, q_0, q), E, q_0, q), \\ k_{0,V,B}(E, q_0, q) &= \dot{x}(0, E, q_0, q), \end{aligned}$$

where we define $s = s_{V,B}(E, q_0, q)$ as the root of the equation

$$x(s, E, q_0, q) = q, \quad s > 0.$$

For $q_0 = q \in \bar{D}$, we put $s_{V,B}(E, q_0, q) = 0$.

Note that

$$\begin{aligned} |k_{0,V,B}(E, q_0, q)| &= c\sqrt{1 - \left(\tfrac{E - V(q_0)}{c^2}\right)^{-2}}, \\ |k_{V,B}(E, q_0, q)| &= c\sqrt{1 - \left(\tfrac{E - V(q)}{c^2}\right)^{-2}}, \end{aligned} \tag{2.4}$$

for $(q, q_0) \in (\bar{D} \times \bar{D}) \backslash \bar{G}$.

Using Properties (2.1) and (2.2), we obtain

Lemma 2.1. *At fixed $E > E(\|V\|_{C^2,D}, \|B\|_{C^1,D}, D)$, we have that $s_{V,B}(E, q_0, q) \in C(\bar{D} \times \bar{D}, \mathbb{R})$, $s_{V,B}(E, q_0, q) \in C^1((\bar{D} \times \bar{D}) \backslash \bar{G}, \mathbb{R})$ and $k_{V,B}(E, q_0, q) \in C^1((\bar{D} \times \bar{D}) \backslash \bar{G}, \mathbb{R}^n)$.*

We consider $k_{V,B}(E, q_0, q)$, $k_{0,V,B}(E, q_0, q)$, $q_0, q \in \partial D$, $q_0 \neq q$ as the boundary value data.

Remark 2.2. Note that if $x(t)$ is solution of (1.1) under condition (1.3), then $x(-t)$ is solution of (1.1) with B replaced by $-B \in \mathcal{F}_{mag}(\bar{D})$ under condition (1.3). Hence the following equalities are valid: at fixed $E > E(\|V\|_{C^2,D}, \|B\|_{C^1,D}, D)$,

$$\begin{aligned} k_{0,V,B}(E, q_0, q) &= -k_{V,-B}(E, q, q_0), && (2.5)\\ k_{V,B}(E, q_0, q) &= -k_{0,V,-B}(E, q, q_0), && (2.6)\\ s_{V,B}(E, q_0, q) &= s_{V,-B}(E, q, q_0), && (2.7) \end{aligned}$$

for $q_0, q \in \bar{D}$, $q_0 \neq q$.

2.2 *Properties of the scattering operator.* For equation (1.1) under condition (1.4), the following is valid (see [Y]): for any $(v_-, x_-) \in B_c \times \mathbb{R}^n$, $v_- \neq 0$, the equation (1.1) under condition (1.4) has a unique solution $x \in C^2(\mathbb{R}, \mathbb{R}^n)$ such that

$$x(t) = v_- t + x_- + y_-(t), \tag{2.8}$$

where $\dot{y}_-(t) \to 0$, $y_-(t) \to 0$, as $t \to -\infty$; in addition for almost any $(v_-, x_-) \in B_c \times \mathbb{R}^n$, $v_- \neq 0$,

$$x(t) = v_+ t + x_+ + y_+(t), \tag{2.9}$$

where $v_+ \neq 0$, $|v_+| < c$, $v_+ = a(v_-, x_-)$, $x_+ = b(v_-, x_-)$, $\dot{y}_+(t) \to 0$, $y_+(t) \to 0$, as $t \to +\infty$.

The map $S : B_c \times \mathbb{R}^n \to B_c \times \mathbb{R}^n$ given by the formulas

$$v_+ = a(v_-, x_-),\ x_+ = b(v_-, x_-) \tag{2.10}$$

is called the scattering map for the equation (1.1) under condition (1.4). The functions $a(v_-, x_-)$, $b(v_-, x_-)$ are called the scattering data for the equation (1.1) under condition (1.4).

By $\mathcal{D}(S)$ we denote the domain of definition of S; by $\mathcal{R}(S)$ we denote the range of S (by definition, if $(v_-, x_-) \in \mathcal{D}(S)$, then $v_- \neq 0$ and $a(v_-, x_-) \neq 0$).

The map S has the following simple properties (see [Y]): $\mathcal{D}(S)$ is an open set of $B_c \times \mathbb{R}^n$ and $\mathrm{Mes}((B_c \times \mathbb{R}^n) \backslash \mathcal{D}(S)) = 0$ for the Lebesgue measure on $B_c \times \mathbb{R}^n$ induced by the Lebesgue measure on $\mathbb{R}^n \times \mathbb{R}^n$; the map $S : \mathcal{D}(S) \to \mathcal{R}(S)$ is continuous and preserves the element of volume; for any $(v, x) \in \mathcal{D}(S)$, $a(v, x)^2 = v^2$.

For $E > c^2$, the map S restricted to

$$\Sigma_E = \{(v_-, x_-) \in B_c \times \mathbb{R}^n \mid |v_-| = c\sqrt{1 - \left(\frac{c^2}{E}\right)^2}\}$$

is the scattering operator at fixed energy E and is denoted by S_E.

We will use the fact that the map S is uniquely determined by its restriction to $\mathcal{M}(S) = \mathcal{D}(S) \cap \mathcal{M}$, where

$$\mathcal{M} = \{(v_-, x_-) \in B_c \times \mathbb{R}^n | v_- \neq 0, v_- x_- = 0\}.$$

This observation is based on the fact that if $x(t)$ satisfies (1.1), then $x(t+t_0)$ also satisfies (1.1) for any $t_0 \in \mathbb{R}$. In particular, the map S at fixed energy E is uniquely determined by its restriction to $\mathcal{M}_E(S) = \mathcal{D}(S) \cap \mathcal{M}_E$, where $\mathcal{M}_E = \Sigma_E \cap \mathcal{M}$.

2.3 *Relation between scattering data and boundary value data.* Assume that

$$V \in C_0^2(\bar{D}, \mathbb{R}),\ B \in C_0^1(\bar{D}, A_n(\mathbb{R})),\ B \in \mathcal{F}_{mag}(\bar{D}). \tag{2.11}$$

We consider equation (1.1) under condition (1.3) and equation (1.1) under condition (1.4). We shall connect the boundary value data $k_{V,B}(E, q_0, q)$, $k_{0,V,B}(E, q_0, q)$ for $E > E(\|V\|_{C^2,D}, \|B\|_{C^1,D}, D)$ and $(q, q_0) \in (\partial D \times \partial D) \backslash \partial G$, to the scattering data a, b.

Proposition 2.1. *Let $E > E(\|V\|_{C^2,D}, \|B\|_{C^1,D}, D)$. Under condition* (2.11), *the following statement is valid: $s_{V,B}(E, q_0, q)$, $k_{V,B}(E, q_0, q)$, $k_{0,V,B}(E, q_0, q)$ given for all $(q, q_0) \in (\partial D \times \partial D) \backslash \partial G$, are determined uniquely by the scattering data $a(v_-, x_-)$, $b(v_-, x_-)$ given for all $(v_-, x_-) \in \mathcal{M}_E(S)$. The converse statement holds: $s_{V,B}(E, q_0, q)$, $k_{V,B}(E, q_0, q)$, $k_{0,V,B}(E, q_0, q)$ given for all $(q, q_0) \in (\partial D \times \partial D) \backslash \partial G$, determine uniquely the scattering data $a(v_-, x_-)$, $b(v_-, x_-)$ for all $(v_-, x_-) \in \mathcal{M}_E(S)$.*

Proof of Proposition 2.1. First of all we introduce functions χ, τ_- and τ_+ dependent on D.

For $(v, x) \in \mathbb{R}^n \backslash \{0\} \times \mathbb{R}^n$, $\chi(v, x)$ denotes the nonnegative number of points contained in the intersection of ∂D with the straight line parametrized by $\mathbb{R} \to \mathbb{R}^n, t \mapsto tv + x$. As D is a strictly convex open subset of $\mathbb{R}^n$, $\chi(v, x) \leq 2$ for all $v, x \in \mathbb{R}^n$, $v \neq 0$.

Let $(v, x) \in \mathbb{R}^n \backslash \{0\} \times \mathbb{R}^n$. Assume that $\chi(v, x) \geq 1$. The real $\tau_-(v, x)$ denotes the smallest real number t such that $\tau_-(v, x)v + x \in \partial D$, and the real $\tau_+(v, x)$ denotes the greatest real number t such that $\tau_+(v, x)v + x \in \partial D$ (if $\chi(v, x) = 1$ then $\tau_-(v, x) = \tau_+(v, x)$).

Direct statement. Let $(q_0, q) \in (\partial D \times \partial D) \backslash \partial G$. Under conditions (2.11) and from (2.1) and (2.2), it follows that there exists an unique $(v_-, x_-) \in \mathcal{M}_E(S)$ such that

$$\begin{aligned}
&\chi(v_-, x_-) = 2,\\
&q_0 = x_- + \tau_-(v_-, x_-)v_-,\\
&q = b(v_-, x_-) + \tau_+(a(v_-, x_-), b(v_-, x_-))a(v_-, x_-).
\end{aligned}$$

In addition, $s_{V,B}(E,q_0,q) = \tau_+(a(v_-,x_-),b(v_-,x_-)) - \tau_-(v_-,x_-)$ and $k_{V,B}(E,q_0,q) = a(v_-,x_-)$ and $k_{0,V,B}(E,q_0,q) = v_-$.
Converse statement. Let $(v_-,x_-) \in \mathcal{M}_E(S)$. Under conditions (2.11), if $\chi(v_-,x_-) \le 1$ then $(a(v_-,x_-),b(v_-,x_-)) = (v_-,x_-)$.

Assume that $\chi(v_-,x_-) = 2$. Let

$$q_0 = x_- + \tau_-(v_-,x_-)v_-.$$

From (2.1) and (2.2) it follows that there is one and only one solution of the equation

$$k_{0,V,B}(E,q_0,q) = v_-,\ \ q \in \partial D,\ \ q \neq q_0. \tag{2.12}$$

We denote by $q(v_-,x_-)$ the unique solution of (2.12). Hence we obtain

$$\begin{aligned} a(v_-,x_-) = \ & k_{V,B}(E,q_0,q(v_-,x_-)),\\ b(v_-,x_-) = \ & q(v_-,x_-) - k_{V,B}(E,q_0,q(v_-,x_-))(s_{V,B}(E,q_0,q(v_-,x_-))\\ & +\tau_-(v_-,x_-)). \end{aligned}$$

Proposition 2.1 is proved. $\square$

For a more complete discussion about connection between scattering data and boundary value data, see [N2] considering the nonrelativistic Newton equation (without magnetic field).

3 Inverse boundary value problem

In this Section, Problem 1 of Introduction is studied.

3.1 *Notations.* For $x \in \bar{D}$, and for $E > V(x) + c^2$, we define

$$r_{V,E}(x) = c\sqrt{\left(\frac{E - V(x)}{c^2}\right)^2 - 1}.$$

At $E > E(\|V\|_{C^2,D}, \|B\|_{C^1,D}, D)$ for $q_0, q \in \bar{D} \times \bar{D}$, $q_0 \neq q$, we define the vectors $\bar{k}_{V,B}(E,q_0,q)$ and $\bar{k}_{0,V,B}(E,q_0,q)$ by

$$\begin{aligned} \bar{k}_{V,B}(E,q_0,q) &= \frac{k_{V,B}(E,q_0,q)}{\sqrt{1 - \frac{|k_{V,B}(E,q_0,q)|^2}{c^2}}},\\ \bar{k}_{0,V,B}(E,q_0,q) &= \frac{k_{0,V,B}(E,q_0,q)}{\sqrt{1 - \frac{|k_{0,V,B}(E,q_0,q)|^2}{c^2}}}. \end{aligned} \tag{3.1}$$

and $\bar{k}_{V,B}(E,q_0,q) = (\bar{k}^1_{V,B}(E,q_0,q),\ldots,\bar{k}^n_{V,B}(E,q_0,q))$, $\bar{k}_{0,V,B}(E,q_0,q) = (\bar{k}^1_{0,V,B}(E,q_0,q),\ldots,\bar{k}^n_{0,V,B}(E,q_0,q))$. Note that from (2.4), it follows that

$$\begin{aligned} |\bar{k}_{V,B}(E,q_0,q)| &= r_{V,E}(q),\\ |\bar{k}_{0,V,B}(E,q_0,q)| &= r_{V,E}(q_0). \end{aligned} \tag{3.2}$$

For $B \in \mathcal{F}_{mag}(\bar{D})$, let $\mathcal{F}_{pot}(D,B)$ be the set of C^1 magnetic potentials for the magnetic field B, i.e.

$$\mathcal{F}_{pot}(D,B) := \{\mathbf{A} \in C^1(\bar{D},\mathbb{R}^n) \mid B_{i,k}(x) = \tfrac{\partial}{\partial x_i}\mathbf{A}_k(x) - \tfrac{\partial}{\partial x_k}\mathbf{A}_i(x),\ x \in \bar{D},\ i,k = 1\ldots n\}. \tag{3.3}$$

(The set $\mathcal{F}_{pot}(D,B)$ is not empty: take, for example, $\mathbf{A}(x) = -\int_0^1 sB(x_0 + s(x-x_0))(x-x_0)ds$, for $x \in \bar{D}$ and some fixed point x_0 of $\bar{D}$.)

3.2 *Hamiltonian mechanics.* Let $\mathbf{A} \in \mathcal{F}_{pot}(D,B)$. The equation (1.1) in D is the Euler-Lagrange equation for the Lagrangian L defined by $L(\dot{x},x) = -c^2\sqrt{1-\frac{\dot{x}^2}{c^2}} + c^{-1}\mathbf{A}(x)\circ\dot{x} - V(x)$, $\dot{x} \in B_c$ and $x \in D$, where $\circ$ denotes the usual scalar product on $\mathbb{R}^n$. The Hamiltonian H associated to the Lagrangian L by Legendre's transform (with respect to $\dot{x}$) is $H(P,x) = c^2\left(1 + c^{-2}|P - c^{-1}\mathbf{A}(x)|^2\right)^{1/2} + V(x)$ where $P \in \mathbb{R}^n$ and $x \in D$. Then equation (1.1) in D is equivalent to the Hamilton's equation

$$\begin{aligned} \dot{x} &= \tfrac{\partial H}{\partial P}(P,x),\\ \dot{P} &= -\tfrac{\partial H}{\partial x}(P,x), \end{aligned} \tag{3.4}$$

for $P \in \mathbb{R}^n$, $x \in D$.

For a solution $x(t)$ of equation (1.1) in D, we define the impulse vector

$$P(t) = \frac{\dot{x}(t)}{\sqrt{1-\frac{|\dot{x}(t)|^2}{c^2}}} + c^{-1}\mathbf{A}(x(t)).$$

Further for $q_0, q \in \bar{D}$, $q_0 \neq q$, and $t \in [0, s(E,q_0,q)]$, we consider

$$P(t,E,q_0,q) = \frac{\dot{x}(t,E,q_0,q)}{\sqrt{1-\frac{|\dot{x}(t,E,q_0,q)|^2}{c^2}}} + c^{-1}\mathbf{A}(x(t,E,q_0,q)), \tag{3.5}$$

where $x(.,E,q_0,q)$ is the solution given by (2.2). From Maupertuis's principle (see [A]), it follows that if $x(t)$, $t \in [t_1,t_2]$, is a solution of (1.1) in D with energy E, then $x(t)$ is a critical point of the functional $\mathcal{A}(y) = \int_{t_1}^{t_2}[r_{V,E}(y(t))|\dot{y}(t)| + c^{-1}\mathbf{A}(y(t))\circ\dot{y}(t)]\,dt$ defined on the set of the functions

$y \in C^1([t_1,t_2],D)$, with boundary conditions $y(t_1)=x(t_1)$ and $y(t_2)=x(t_2)$. Note that for $q_0, q \in D$, $q_0 \neq q$, functional $\mathcal{A}$ taken along the trajectory of the solution $x(.,E,q_0,q)$ given by (2.2) is equal to the reduced action $\mathcal{S}_{0_{V,\mathbf{A},E}}(q_0,q)$ from q_0 to q at fixed energy E for (3.4), where

$$\mathcal{S}_{0_{V,\mathbf{A},E}}(q_0,q) = \begin{cases} 0, & \text{if } q_0 = q, \\ \int_0^{s(E,q_0,q)} P(s,E,q_0,q)\circ \dot{x}(s,E,q_0,q)ds, & \text{if } q_0 \neq q, \end{cases} \tag{3.6}$$

for $q_0, q \in \bar{D}$.

3.3 *Properties of $\mathcal{S}_{0_{V,\mathbf{A},E}}$ at fixed and sufficiently large energy E.* The following Propositions 3.1, 3.2 give properties of $\mathcal{S}_{0_{V,\mathbf{A},E}}$ at fixed and sufficiently large energy E.

Proposition 3.1. *Let $E > E(\|V\|_{C^2,D}, \|B\|_{C^1,D}, D)$. The following statements are valid:*

$$\mathcal{S}_{0_{V,\mathbf{A},E}} \in C(\bar{D}\times\bar{D},\mathbb{R}), \tag{3.7}$$

$$\mathcal{S}_{0_{V,\mathbf{A},E}} \in C^2((\bar{D}\times\bar{D})\backslash\bar{G},\mathbb{R}), \tag{3.8}$$

$$\frac{\partial \mathcal{S}_{0_{V,\mathbf{A},E}}}{\partial x_i}(\zeta,x) = \bar{k}^i_{V,B}(E,\zeta,x) + c^{-1}\mathbf{A}_i(x), \tag{3.9}$$

$$\frac{\partial \mathcal{S}_{0_{V,\mathbf{A},E}}}{\partial \zeta_i}(\zeta,x) = -\bar{k}^i_{0,V,B}(E,\zeta,x) - c^{-1}\mathbf{A}_i(\zeta), \tag{3.10}$$

$$\frac{\partial^2 \mathcal{S}_{0_{V,\mathbf{A},E}}}{\partial \zeta_i \partial x_j}(\zeta,x) = -\frac{\partial \bar{k}^i_{0,V,B}}{\partial x_j}(E,\zeta,x) = \frac{\partial \bar{k}^j_{V,B}}{\partial \zeta_i}(E,\zeta,x), \tag{3.11}$$

for $(\zeta,x) \in (\bar{D}\times\bar{D})\backslash\bar{G}$, $\zeta=(\zeta_1,..,\zeta_n)$, $x=(x_1,..,x_n)$, and $i,j=1\ldots n$. In addition,

$$\max\left(\left|\frac{\partial \mathcal{S}_{0_{V,\mathbf{A},E}}}{\partial x_i}(\zeta,x)\right|, \left|\frac{\partial \mathcal{S}_{0_{V,\mathbf{A},E}}}{\partial \zeta_i}(\zeta,x)\right|\right) \le M_1, \tag{3.12}$$

$$\left|\frac{\partial^2 \mathcal{S}_{0_{V,\mathbf{A},E}}}{\partial \zeta_i \partial x_j}(\zeta,x)\right| \le \frac{M_2}{|\zeta - x|}, \tag{3.13}$$

for $(\zeta,x) \in (\bar{D}\times\bar{D})\backslash\bar{G}$, $\zeta=(\zeta_1,..,\zeta_n)$, $x=(x_1,..,x_n)$, and $i,j=1\ldots n$, and where M_1 and M_2 depend on V, B and D.

Proposition 3.1 is proved in Section 5.

Equalities (3.9) and (3.10) are known formulas of classical Hamiltonian mechanics (see Section 46 and further Sections of [A]).

Proposition 3.2. *Let $E > E(\|V\|_{C^2,D}, \|B\|_{C^1,D}, D)$. The map $\nu_{V,B,E}$: $\partial D \times D \to \mathbb{S}^{n-1}$, defined by*

$$\nu_{V,B,E}(\zeta, x) = -\frac{k_{V,B}(E, \zeta, x)}{|k_{V,B}(E, \zeta, x)|}, \text{ for } (\zeta, x) \in \partial D \times D, \tag{3.14}$$

has the following properties:

$$\begin{array}{l} \nu_{V,B,E} \in C^1(\partial D \times D, \mathbb{S}^{n-1}), \\ \text{the map } \nu_{V,B,E,x} : \partial D \to \mathbb{S}^{n-1},\ \zeta \mapsto \nu_{V,B,E}(\zeta, x), \text{ is a} \\ C^1 \text{ orientation preserving diffeomorphism from } \partial D \text{ onto } \mathbb{S}^{n-1} \end{array} \tag{3.15}$$

for $x \in D$ (where we choose the canonical orientation of $\mathbb{S}^{n-1}$ and the orientation of ∂D given by the canonical orientation of $\mathbb{R}^n$ and the unit outward normal vector).

Proposition 3.2 follows from (1.1), (1.2) and properties (2.1), (2.2).

Remark 3.1. Taking account of (3.9) and (3.10), we obtain the following formulas: at $E > E(\|V\|_{C^2,D}, \|B\|_{C^1,D}, D)$, for any $x,\ \zeta \in \bar{D}$, $x \neq \zeta$,

$$B_{i,j}(x) = -c\left(\frac{\partial \bar{k}^j_{V,B}}{\partial x_i}(E, \zeta, x) - \frac{\partial \bar{k}^i_{V,B}}{\partial x_j}(E, \zeta, x)\right), \tag{3.16}$$

$$B_{i,j}(x) = -c\left(\frac{\partial \bar{k}^j_{0,V,B}}{\partial x_i}(E, x, \zeta) - \frac{\partial \bar{k}^i_{0,V,B}}{\partial x_j}(E, x, \zeta)\right). \tag{3.17}$$

3.4 *Results of uniqueness.* We denote by $\omega_{0,V,B}$ the $n-1$ differential form on $\partial D \times D$ obtained in the following manner:

- for $x \in D$, let $\omega_{V,B,x}$ be the pull-back of ω_0 by $\nu_{V,B,E,x}$ where ω_0 denotes the canonical orientation form on $\mathbb{S}^{n-1}$ (i.e. $\omega_0(\zeta)(v_1, .., v_{n-1}) = \det(\zeta, v_1, .., v_{n-1})$, for $\zeta \in \mathbb{S}^{n-1}$ and $v_1, .., v_{n-1} \in T_\zeta\mathbb{S}^{n-1}$),

- for $(\zeta, x) \in \partial D \times D$ and for any $v_1, .., v_{n-1} \in T_{(\zeta,x)}(\partial D \times D)$,

$$\omega_{0,V,B}(\zeta, x)(v_1, .., v_{n-1}) = \omega_{V,B,x}(\zeta)(\sigma'_{(\zeta,x)}(v_1), .., \sigma'_{(\zeta,x)}(v_{n-1})),$$

where $\sigma : \partial D \times D \to \partial D$, $(\zeta', x') \mapsto \zeta'$, and $\sigma'_{(\zeta,x)}$ denotes the derivative (linear part) of σ at (ζ, x).

From smoothness of $\nu_{V,B,E}$, σ and ω_0, it follows that $\omega_{0,V,B}$ is a continuous $n-1$ form on $\partial D \times D$.

Now let $\lambda \in \mathbb{R}^+$ and $V_1, V_2 \in C^2(\bar{D}, \mathbb{R})$, $B_1, B_2 \in \mathcal{F}_{mag}(\bar{D})$, such that $\max(\|V_1\|_{C^2,D}, \|V_2\|_{C^2,D}, \|B_1\|_{C^1,D}, \|B_2\|_{C^1,D}) \leq \lambda$. For $\mu = 1, 2$, let $\mathbf{A}_\mu \in \mathcal{F}_{pot}(D, B_\mu)$.

Let $E > E(\lambda, \lambda, D)$ where $E(\lambda, \lambda, D)$ is defined in (1.7). Consider β^1, β^2 the differential one forms defined on $(\partial D \times \bar{D})\backslash\bar{G}$ by

$$\beta^\mu(\zeta, x) = \sum_{j=1}^{n} \bar{k}^j_{V_\mu, B_\mu}(E, \zeta, x)dx_j, \tag{3.18}$$

for $(\zeta, x) \in (\partial D \times \bar{D})\backslash\bar{G}$, $x = (x_1, \ldots, x_n)$ and $\mu = 1, 2$.

Consider the differential forms Φ_0 on $(\partial D\times\bar{D})\backslash\bar{G}$ and Φ_1 on $(\partial D\times\bar{D})\backslash\bar{G}$ defined by

$$\begin{aligned}\Phi_0(\zeta, x) \;&=\; -(-1)^{\frac{n(n+1)}{2}}(\beta^2 - \beta^1)(\zeta, x) \wedge d_\zeta(\mathcal{S}_{0_{V_2,\mathbf{A}_2,E}} - \mathcal{S}_{0_{V_1,\mathbf{A}_1,E}})(\zeta, x)\\ &\wedge \sum_{p+q=n-2} (dd_\zeta \mathcal{S}_{0_{V_1,\mathbf{A}_1,E}}(\zeta, x))^p \wedge (dd_\zeta \mathcal{S}_{0_{V_2,\mathbf{A}_2,E}}(\zeta, x))^q, \end{aligned}\tag{3.19}$$

for $(\zeta, x) \in (\partial D \times \bar{D})\backslash\bar{G}$, where $d = d_\zeta + d_x$,

$$\begin{aligned}\Phi_1(\zeta, x) \;&=\; -(-1)^{\frac{n(n-1)}{2}}\big[\beta^1(\zeta, x) \wedge (dd_\zeta \mathcal{S}_{0_{V_1,\mathbf{A}_1,E}}(\zeta, x))^{n-1}\\ &+\beta^2(\zeta, x) \wedge (dd_\zeta \mathcal{S}_{0_{V_2,\mathbf{A}_2,E}}(\zeta, x))^{n-1} - \beta^1(\zeta, x)\\ &\wedge(dd_\zeta \mathcal{S}_{0_{V_2,\mathbf{A}_2,E}}(\zeta, x))^{n-1} - \beta^2(\zeta, x) \wedge (dd_\zeta \mathcal{S}_{0_{V_1,\mathbf{A}_1,E}}(\zeta, x))^{n-1}\big],\end{aligned}\tag{3.20}$$

for $(\zeta, x) \in (\partial D \times \bar{D})\backslash\bar{G}$, where $d = d_\zeta + d_x$.

Consider the C^2 map $incl : (\partial D \times \partial D)\backslash\partial G \to (\partial D \times \bar{D})\backslash\bar{G}$, $(\zeta, x) \mapsto (\zeta, x)$.

From (3.8), (3.12) and (3.13), it follows that Φ_0 is continuous on $(\partial D \times \bar{D})\backslash\bar{G}$ and $incl^*(\Phi_0)$ is integrable on $\partial D\times\partial D$ and Φ_1 is continuous on $(\partial D \times \bar{D})\backslash\bar{G}$ and integrable on $\partial D \times \bar{D}$ (where $incl^*(\Phi_0)$ is the pull-back of the differential form Φ_0 by the inclusion map $incl$).

Lemma 3.1. *Let $\lambda \in \mathbb{R}^+$ and $E > E(\lambda, \lambda, D)$. Let $V_1, V_2 \in C^2(\bar{D}, \mathbb{R})$, $B_1, B_2 \in \mathcal{F}_{mag}(\bar{D})$ such that* $\max(\|V_1\|_{C^2,D}, \|V_2\|_{C^2,D}, \|B_1\|_{C^1,D}, \|B_2\|_{C^1,D}) \le \lambda$. *For $\mu = 1, 2$, let $\mathbf{A}_\mu \in \mathcal{F}_{pot}(D, B_\mu)$. The following equalities are valid:*

$$\int_{\partial D\times\partial D} incl^*(\Phi_0) = \int_{\partial D\times\bar{D}} \Phi_1; \tag{3.21}$$

$$\begin{aligned}\frac{1}{(n-1)!}\Phi_1(\zeta, x) \;&=\; (r_{V_1,E}(x)^n\omega_{0,V_1,B_1}(\zeta, x) + r_{V_2,E}(x)^n\omega_{0,V_2,B_2}(\zeta, x)\\ &-\bar{k}_{V_1,B_1}(E, \zeta, x) \circ \bar{k}_{V_2,B_2}(E, \zeta, x)\\ &\times \big(r_{V_1,E}(x)^{n-2}\omega_{0,V_1,B_1}(\zeta, x) + r_{V_2,E}(x)^{n-2}\\ &\times\omega_{0,V_2,B_2}(\zeta, x))) \wedge dx_1 \wedge .. \wedge dx_n,\end{aligned}\tag{3.22}$$

for $(\zeta, x) \in \partial D \times D$.

Equality (3.21) follows from regularization and Stokes' formula. Proof of Lemma 3.1 is given in Section 4.

Taking account of Lemma 3.1, Proposition 3.2 and Remark 3.1, we obtain the following Theorem of uniqueness.

Theorem 3.1. *Let* $\lambda \in \mathbb{R}^+$ *and* $E > E(\lambda, \lambda, D)$. *Let* $V_1, V_2 \in C^2(\bar{D}, \mathbb{R})$, $B_1, B_2 \in \mathcal{F}_{mag}(\bar{D})$ *such that* $\max(\|V_1\|_{C^2,D}, \|V_2\|_{C^2,D}, \|B_1\|_{C^1,D}, \|B_2\|_{C^1,D}) \leq \lambda$. *For* $\mu = 1, 2$, *let* $\mathbf{A}_\mu \in \mathcal{F}_{pot}(D, B_\mu)$. *The following estimate is valid:*

$$\int_D \left(r_{V_1,E}(x) - r_{V_2,E}(x)\right)\left(r_{V_1,E}(x)^{n-1} - r_{V_2,E}(x)^{n-1}\right) dx \leq$$
$$\frac{\Gamma(\frac{n}{2})}{2\pi^{\frac{n}{2}}(n-1)!} \int_{\partial D \times \partial D} incl^*(\Phi_0). \tag{3.23}$$

In addition, the following statements are valid:

if $k_{V_1,B_1}(E, \zeta, x) = k_{V_2,B_2}(E, \zeta, x)$ *for* $\zeta, x \in \partial D$, $\zeta \neq x$, *then* $V_1 \equiv V_2$ *and* $B_1 \equiv B_2$ *on* $\bar{D}$; *if* $k_{0,V_1,B_1}(E, \zeta, x) = k_{0,V_2,B_2}(E, \zeta, x)$ *for* $\zeta, x \in \partial D$, $\zeta \neq x$, *then* $V_1 \equiv V_2$ *and* $B_1 \equiv B_2$ *on* $\bar{D}$.

Proof of Theorem 3.1 is given in Section 4.

If $B_1 \equiv 0$, $B_2 \equiv 0$ and V_1, V_2, and D are smoother than C^2, then Lemma 3.1 and Theorem 3.1 follow from results of [B] and [GN].

4 Proof of Theorem 3.1 and Lemma 3.1

Using Lemma 2.1, (2.4), Propositions 3.1, 3.2 and Lemma 3.1, we first prove Theorem 3.1.

Proof of Theorem 3.1. From (3.22) and Proposition 3.2 and definition of ω_{0,V_μ,B_μ}, $\mu = 1, 2$, it follows that

$$\frac{1}{(n-1)!} \int_{\partial D \times \bar{D}} \Phi_1 = \tag{4.1}$$
$$\int_D r_{V_1,E}(x)^n \int_{\mathbb{S}^{n-1}} \left(1 + \frac{w \circ \bar{k}_{V_2,B_2}(E, \zeta_{1,x}(w), x)}{r_{V_1,E}(x)}\right) d\sigma(w) dx$$
$$+ \int_D r_{V_2,E}(x)^n \int_{\mathbb{S}^{n-1}} \left(1 + \frac{\bar{k}_{V_1,B_1}(E, \zeta_{2,x}(w), x) \circ w}{r_{V_2,E}(x)}\right) d\sigma(w) dx,$$

where $d\sigma$ is the canonical measure on $\mathbb{S}^{n-1}$, and where $\circ$ denotes the usual scalar product on $\mathbb{R}^n$, and where for $x \in D$ and $w \in \mathbb{S}^{n-1}$ and $\mu = 1, 2$, $\zeta_{\mu,x}(w)$ denotes the unique point ζ of ∂D such that $w = \nu_{V_\mu,B_\mu,E,x}(\zeta)$. Hence using Cauchy-Bunyakovski-Schwarz inequality and (3.2) and the equality $\int_{\mathbb{S}^{n-1}} d\sigma(w) = \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}$, we obtain

$$\frac{1}{(n-1)!}\int_{\partial D\times\bar{D}}\Phi_1 \geq \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}\int_D (r_{V_1,E}(x)-r_{V_2,E}(x))(r_{V_1,E}(x)^{n-1}-r_{V_2,E}(x)^{n-1})dx. \tag{4.2}$$

Estimate (4.2) and equality (3.21) prove (3.23).

Now assume that $k_{V_1,B_1}(E,\zeta,x) = k_{V_2,B_2}(E,\zeta,x)$ for $\zeta, x\in\partial D$, $\zeta\neq x$. Then from (3.1) and (3.18), it follows that the one form $incl^*(\beta^2-\beta^1)(\zeta,x)$ is null for any $\zeta,x\in\partial D$, $\zeta\neq x$. Hence from (3.19), it follows that the $2n-2$ form $incl^*(\Phi_0)(\zeta,x)$ is null for any $\zeta,x\in\partial D,\ \zeta\neq x$. Thus using (3.23), we obtain $\int_D (r_{V_1,E}(x)-r_{V_2,E}(x))(r_{V_1,E}(x)^{n-1}-r_{V_2,E}(x)^{n-1})dx\leq 0$, and as $n\geq 2$, this latter inequality implies that

$$r_{V_1,E}\equiv r_{V_2,E} \text{ on } \bar{D}. \tag{4.3}$$

Thus $V_1\equiv V_2$.

Using (4.3) and the equality $|\bar{k}_{V_i,B_i}(E,\zeta,x)| = r_{V_i,E}(x)$ for $i=1,2$, $x\in D$ and $\zeta\in\partial D$, and using (4.1), we obtain that

$$\begin{aligned}\frac{1}{(n-1)!}\int_{\partial D\times\bar{D}}\Phi_1 &= \frac{1}{2}\sum_{i=1}^{2}\int_D r_{V_i,E}(x)^{n-2}\int_{\mathbb{S}^{n-1}}\big|\bar{k}_{V_1,B_1}(E,\zeta_{i,x}(w),x)\\ &\quad -\bar{k}_{V_2,B_2}(E,\zeta_{i,x}(w),x)\big|^2 d\sigma(w)dx.\end{aligned}$$

As $\int_{\partial D\times\bar{D}}\Phi_1 = 0$ (due to (3.21)), we obtain that for any $x\in D$, and any $w\in\mathbb{S}^{n-1}$, $\bar{k}_{V_1,B_1}(E,\zeta_{1,x}(w),x) = \bar{k}_{V_2,B_2}(E,\zeta_{1,x}(w),x)$. At fixed $x\in D$, we know that $\zeta_{1,x}$ is onto ∂D. Hence the following equality is valid

$$\bar{k}_{V_1,B_1}(E,\zeta,x) = \bar{k}_{V_2,B_2}(E,\zeta,x),\ \zeta\in\partial D,\ x\in D. \tag{4.4}$$

From (4.4) and (3.16), it follows that $B_1\equiv B_2$ on D.

Now assume that $k_{0,V_1,B_1}(E,\zeta,x) = k_{0,V_2,B_2}(E,\zeta,x)$ for $\zeta,x\in\partial D$, $\zeta\neq x$. Then using (2.5) and replacing B_i by $-B_i$, $i=1,2$, in the proof, we obtain $(V_1,B_1)\equiv(V_2,B_2)$. $\square$

Using Lemma 2.1, (2.4), Propositions 3.1, 3.2, we prove Lemma 3.1.

Proof of Lemma 3.1. We first prove (3.22). Let U be an open subset of $\mathbb{R}^{n-1}$ and $\phi: U\to\partial D$ such that ϕ is a C^2 parametrization of ∂D. Let $\phi_0: U\times D\to \partial D\times D$, $(\zeta,x)\mapsto(\phi(\zeta),x)$. We work in coordinates given by $(U\times D,\phi_0)$. Let

$\mu, \mu' = 1, 2$. On one hand from definition of ω_{0,V_μ,B_μ}, definition of $\nu_{V_\mu,B_\mu,E,x}$ and (3.2), we obtain

$$\begin{aligned}&\omega_{0,V_\mu,B_\mu}(\zeta,x)\wedge dx_1\wedge\ldots\wedge dx_n = (-1)^n r_{V_\mu,E}(x)^{-n} \\ &\times\det\left(\bar{k}_{V_\mu,B_\mu}(E,\zeta,x), \frac{\partial\bar{k}_{V_\mu,B_\mu}}{\partial\zeta_1}(E,\zeta,x),\ldots,\frac{\partial\bar{k}_{V_\mu,B_\mu}}{\partial\zeta_{n-1}}(E,\zeta,x)\right) \\ &d\zeta_1\wedge\ldots\wedge d\zeta_{n-1}\wedge dx_1\wedge\ldots\wedge dx_n\end{aligned} \tag{4.5}$$

for $\zeta \in U$ and $x \in D$ and $\mu = 1, 2$.

On the other hand straightforward calculations give ($dd_\zeta\mathcal{S}_{0_{V_\mu,\mathbf{A}_\mu,E}}(\zeta,x) = \sum_{\substack{m_1=1..n \\ m_2=1..n-1}} \frac{\partial^2\mathcal{S}_{0_{V_\mu,\mathbf{A}_\mu,E}}}{\partial x_{m_1}\partial\zeta_{m_2}}(\zeta,x)dx_{m_1}\wedge d\zeta_{m_2}$)

$$\begin{aligned}&\beta^{\mu'}(\zeta,x)\wedge(dd_\zeta\mathcal{S}_{0_{V_\mu,\mathbf{A}_\mu,E}}(\zeta,x))^{n-1} = (-1)^{\frac{(n-1)(n-2)}{2}}(n-1)! \\ &\times\det\left(\bar{k}_{V_{\mu'},B_{\mu'}}(E,\zeta,x), \frac{\partial^2\mathcal{S}_{0_{V_\mu,\mathbf{A}_\mu,E}}}{\partial\zeta_1\partial x}(\zeta,x),\ldots,\frac{\partial^2\mathcal{S}_{0_{V_\mu,\mathbf{A}_\mu,E}}}{\partial\zeta_{n-1}\partial x}(\zeta,x)\right) \\ &d\zeta_1\wedge\ldots\wedge d\zeta_{n-1}\wedge dx_1\wedge\ldots\wedge dx_n,\end{aligned} \tag{4.6}$$

for $\zeta \in U$, $x \in D$. Note that due to (3.2) and Proposition 3.2, $\bar{k}_{V_\mu,B_\mu}(E,\zeta,x)$ is orthogonal to $\frac{\partial}{\partial\zeta_m}\bar{k}_{V_\mu,B_\mu}(E,\zeta,x)$, $m = 1\ldots n-1$, and that ($\bar{k}_{V_\mu,B_\mu}(E,\zeta,x)$, $\frac{\partial}{\partial\zeta_1}\bar{k}_{V_\mu,B_\mu}(E,\zeta,x),\ldots,\frac{\partial}{\partial\zeta_{n-1}}\bar{k}_{V_\mu,B_\mu}(E,\zeta,x)$) is a basis of $\mathbb{R}^n$. Hence from (4.5), (4.6) and (3.9), we obtain

$$\begin{aligned}&\beta^{\mu'}(\zeta,x)\wedge(dd_\zeta\mathcal{S}_{0_{V_\mu,\mathbf{A}_\mu,E}}(\zeta,x))^{n-1} = -(-1)^{\frac{n(n-1)}{2}}(n-1)!r_{V_\mu,E}(x)^n \\ &\times\frac{\bar{k}_{V_{\mu'},B_{\mu'}}(E,\zeta,x)\circ\bar{k}_{V_\mu,B_\mu}(E,\zeta,x)}{r_{V_\mu,E}(x)^2}\omega_{0,V_\mu,B_\mu}(\zeta,x)\wedge dx_1\wedge\ldots\wedge dx_n,\end{aligned} \tag{4.7}$$

for $\zeta \in U$, $x \in D$. Definition (3.20) and equality (4.7) proves (3.22).

We sketch the proof of (3.21). Let $\varepsilon \in]0,1[$ and $x_0 \in D$. We consider

$$D_\varepsilon = \{x_0 + \varepsilon(x - x_0) \mid x \in D\}. \tag{4.8}$$

As D is a strictly convex (in the strong sense) open domain of $\mathbb{R}^n$, with C^2 boundary, it follows that D_ε is also a strictly convex (in the strong sense) open domain of $\mathbb{R}^n$, with C^2 boundary, and in addition, as $0 < \varepsilon < 1$,

$$\bar{D}_\varepsilon \subseteq D, \tag{4.9}$$

$$dist(\partial D, \partial D_\varepsilon) = (1-\varepsilon)dist(\partial D, x_0). \tag{4.10}$$

where $dist(\partial D, \partial D_\varepsilon) = \inf\{|x-y| \mid x \in D_\varepsilon,\ y \in \partial D\}$ and $dist(\partial D, x_0) = \inf\{|y - x_0| \mid y \in \partial D\} > 0$.

For $\mathcal{M}$ and $\mathcal{N}$ two finite-dimensional oriented C^2 manifold (with or without boundary), we consider on $\mathcal{M}\times\mathcal{N}$ the differential product structure induced by the already given differential structures of $\mathcal{M}$ and $\mathcal{N}$ and we consider the orientation of $\mathcal{M}\times\mathcal{N}$ given by the already fixed orientation of $\mathcal{M}$ and of $\mathcal{N}$. The orientation of ∂D_ε is given by the unit outward normal vector and $\partial D\times\bar{D}_\varepsilon$ is a C^2 manifold with boundary $\partial D\times\partial D_\varepsilon$ (which is a C^2 manifold without boundary). Let $incl_\varepsilon\in C^2(\partial D\times\partial D_\varepsilon,\partial D\times\bar{D}_\varepsilon)$ be defined by $incl_\varepsilon(\zeta,x)=(\zeta,x)$, $(\zeta,x)\in\partial D\times\partial D_\varepsilon$. Here we omit the details of the proof of the following statement: $\int_{\partial D\times\bar{D}_\varepsilon}\Phi_1\to\int_{\partial D\times\bar{D}}\Phi_1$ and $\int_{\partial D\times\partial D_\varepsilon} incl_\varepsilon^*(\Phi_0)\to\int_{\partial D\times\partial D} incl^*(\Phi_0)$ as $\varepsilon\to 1^-$. These statements follow from (3.12), (3.13) and (3.9). We shall prove that

$$\int_{\partial D\times\bar{D}_\varepsilon}\Phi_1=\int_{\partial D\times\partial D_\varepsilon} incl_\varepsilon^*(\Phi_0). \tag{4.11}$$

For $\mu=1,2$, let $\mathcal{S}_{0_\mu}\in C^2((\mathbb{R}^n\times\mathbb{R}^n)\backslash\{(x,x)\mid x\in\mathbb{R}^n\},\mathbb{R})$ such that $\mathcal{S}_{0_\mu}(\zeta,x)=\mathcal{S}_{0_{V_\mu,\mathbf{A}_\mu,E}}(\zeta,x)$, $(\zeta,x)\in(\bar{D}\times\bar{D})\backslash\bar{G}$ (from (3.8), it follows that such a function $\mathcal{S}_{0_\mu}$ exists). Let $\delta_\varepsilon=dist(\partial D,\partial D_\varepsilon)$. Let W_{1,δ_ε} be the open subset $\partial D+B(0,\frac{\delta_\varepsilon}{2})=\{x+y\mid x\in\partial D,\ y\in\mathbb{R}^n,\ |y|<\frac{\delta_\varepsilon}{2}\}$ and let W_{2,δ_ε} be the open subset $D_\varepsilon+B(0,\frac{\delta_\varepsilon}{2})=\{x+y\mid x\in D_\varepsilon,\ y\in\mathbb{R}^n,\ |y|<\frac{\delta_\varepsilon}{2}\}$. Note that W_{1,δ_ε} is an open neighborhood of ∂D which does not intersect W_{2,δ_ε} which is an open neighborhood of $\bar{D}_\varepsilon$. Hence $\mathcal{S}_{0_\mu}\in C^2(W_{1,\delta_\varepsilon}\times W_{2,\delta_\varepsilon},\mathbb{R})$ and there exists a sequence of functions $(\mathcal{S}_{0_{\mu,m}})$ such that

$$\mathcal{S}_{0_{\mu,m}}\in C^3(W_{1,\delta_\varepsilon}\times W_{2,\delta_\varepsilon},\mathbb{R}), \tag{4.12}$$

$$\sup_{\substack{(x,y)\in\partial D\times\bar{D}_\varepsilon\\ \alpha=(\alpha_1,\alpha_2)\in\mathbb{N}^2\\ |\alpha|=\alpha_1+\alpha_2\le 2}}\left|\frac{\partial^{|\alpha|}(\mathcal{S}_{0_{\mu,m}}-\mathcal{S}_{0_\mu})}{\partial x_{\alpha_1}\partial y_{\alpha_2}}(x,y)\right|\to 0,\ \text{as } m\to+\infty. \tag{4.13}$$

Fix $m\in\mathbb{N}$. Let $\mu=1,2$. We define the differential one-form, β_m^μ on $(\partial D\times\bar{D}_\varepsilon)$ by

$$\beta_m^\mu(\zeta,x) = d_x\mathcal{S}_{0_{\mu,m}}(\zeta,x)-\frac{1}{c}\sum_{j=1}^n\mathbf{A}_\mu^j(x)dx_j, \tag{4.14}$$

for $(\zeta,x)\in\partial D\times\bar{D}_\varepsilon$ and $x=(x_1,\dots,x_n)$ and $\mathbf{A}_\mu(x)=(\mathbf{A}_\mu^1(x),\dots,\mathbf{A}_\mu^n(x))$ and where $d=d_\zeta+d_x$ is the De Rham differential operator on $\partial D\times\bar{D}_\varepsilon$.

We define the continuous differential $2n-2$ form $\Phi_{0,m}$ on $\partial D\times\bar{D}_\varepsilon$ by

$$\begin{aligned}\Phi_{0,m}(\zeta,x) &= -(-1)^{\frac{n(n+1)}{2}}(\beta_m^2-\beta_m^1)(\zeta,x)\wedge d_\zeta(\mathcal{S}_{0_{2,m}}-\mathcal{S}_{0_{1,m}})(\zeta,x)\\ &\qquad\wedge\sum_{p+q=n-2}(dd_\zeta\mathcal{S}_{0_{1,m}}(\zeta,x))^p\wedge(dd_\zeta\mathcal{S}_{0_{2,m}}(\zeta,x))^q,\end{aligned} \tag{4.15}$$

for $\zeta \in \partial D$, $x \in \bar{D}_\varepsilon$, where $d = d_\zeta + d_x$ is the De Rham differential operator on $\partial D \times \bar{D}_\varepsilon$. We define the continuous differential $2n-1$ form $\Phi_{1,m}$ on $\partial D \times \bar{D}_\varepsilon$ by

$$\begin{aligned}\Phi_{1,m}(\zeta, x) \;\; = \;\; & -(-1)^{\frac{n(n-1)}{2}} \left[\beta_m^1(\zeta, x) \wedge (dd_\zeta \mathcal{S}_{0_{1,m}}(\zeta, x))^{n-1} + \beta_m^2(\zeta, x)\right. \\ & \wedge (dd_\zeta \mathcal{S}_{0_{2,m}}(\zeta, x))^{n-1} - \beta_m^1(\zeta, x) \wedge (dd_\zeta \mathcal{S}_{0_{2,m}}(\zeta, x))^{n-1} \\ & \left. -\beta_m^2(\zeta, x) \wedge (dd_\zeta \mathcal{S}_{0_{1,m}}(\zeta, x))^{n-1}\right], \qquad (4.16)\end{aligned}$$

for $(\zeta, x) \in \partial D \times \bar{D}_\varepsilon$.

From (4.14)-(4.16), (3.9), (4.13), it follows that

$$\int_{\partial D \times \bar{D}_\varepsilon} \Phi_{1,m} \;\; \to \;\; \int_{\partial D \times \bar{D}_\varepsilon} \Phi_1, \quad \text{as } m \to +\infty, \qquad (4.17)$$

$$\int_{\partial D \times \partial D_\varepsilon} incl_\varepsilon^*(\Phi_{0,m}) \;\; \to \;\; \int_{\partial D \times \partial D_\varepsilon} incl_\varepsilon^*(\Phi_0), \quad \text{as } m \to +\infty. \qquad (4.18)$$

If we prove that $\int_{\partial D \times \bar{D}_\varepsilon} \Phi_{1,m} = \int_{\partial D \times \partial D_\varepsilon} incl_\varepsilon^*(\Phi_{0,m})$, then formula (4.11) will follow from (4.17) and (4.18).

From (4.12), it follows that

$$dd_\zeta \mathcal{S}_{0_{\mu,m}} \text{ is a } C^1 \text{ form on } \partial D \times \bar{D}_\varepsilon, \qquad (4.19)$$

$$d(dd_\zeta \mathcal{S}_{0_{\mu,m}}) = 0, \qquad (4.20)$$

where d is the De Rham differential operator on $\partial D \times \bar{D}_\varepsilon$. From (4.14), it follows that

$$d\beta_m^\mu(\zeta, x) = -dd_\zeta \mathcal{S}_{0_{\mu,m}}(\zeta, x) - \frac{1}{c} \sum_{1 \le j_1 < j_2 \le n} B_{j_1,j_2}^\mu(x) dx_{j_1} \wedge dx_{j_2}, \qquad (4.21)$$

for $(\zeta, x) \in \partial D \times \bar{D}_\varepsilon$ and $x = (x_1, \ldots, x_n)$ ($B_{j_1,j_2}^\mu(x)$ denotes the elements of $B_\mu(x)$). From (4.19)-(4.21), it follows that $\Phi_{0,m}$ is C^1 on $\partial D \times \bar{D}_\varepsilon$ and that

$$\begin{aligned}& d\Phi_{0,m}(\zeta, x) = (-1)^{n-1} \Phi_{1,m}(\zeta, x) + \omega(\zeta, x), \qquad (4.22) \\ & \omega(\zeta, x) = (-1)^{\frac{n(n+1)}{2}} \frac{1}{c} \sum_{1 \le j_1 < j_2 \le n} (B_{j_1,j_2}^2(x) - B_{j_1,j_2}^1(x)) dx_{j_1} \wedge dx_{j_2} \\ & \wedge d_\zeta(\mathcal{S}_{0_{2,m}} - \mathcal{S}_{0_{1,m}})(\zeta, x) \wedge \sum_{p+q=n-2} (dd_\zeta \mathcal{S}_{0_{1,m}}(\zeta, x))^p \wedge (dd_\zeta \mathcal{S}_{0_{2,m}}(\zeta, x))^q,\end{aligned}$$

for $(\zeta, x) \in \partial D \times \bar{D}_\varepsilon$ and $x = (x_1, \ldots, x_n)$. Note that as B_μ, $\mu = 1, 2$, is continuously differentiable on $\bar{D}$, then the 2-form defined on $\partial D \times \bar{D}_\varepsilon$ by

$\sum_{1\le j_1<j_2\le n}(B^2_{j_1,j_2}(x) - B^1_{j_1,j_2}(x))dx_{j_1} \wedge dx_{j_2}$, $(\zeta, x) \in \partial D \times \bar{D}_\varepsilon$ is also C^1. Note that ($\bar{D}_\varepsilon$ is a n-dimensional C^2 manifold and (4.20))

$$\omega(\zeta, x) = d\tilde{\omega}(\zeta, x), \tag{4.23}$$

where

$$\tilde{\omega}(\zeta, x) = (-1)^{\frac{n(n+1)}{2}} \frac{(\mathcal{S}_{0_{2,m}} - \mathcal{S}_{0_{1,m}})(\zeta, x)}{c} \sum_{1\le j_1<j_2\le n} (B^2_{j_1,j_2}(x) - B^1_{j_1,j_2}(x))$$
$$dx_{j_1} \wedge dx_{j_2} \wedge \sum_{p+q=n-2} (dd_\zeta \mathcal{S}_{0_{1,m}}(\zeta, x))^p \wedge (dd_\zeta \mathcal{S}_{0_{2,m}}(\zeta, x))^q, \tag{4.24}$$

for $(\zeta, x) \in \partial D \times \bar{D}_\varepsilon$ and $x = (x_1, \ldots, x_n)$. Since ∂D_ε is a $(n-1)$-dimensional C^2 manifold, it follows that

$$incl^*_\varepsilon \tilde{\omega}(\zeta, x) = 0 \tag{4.25}$$

for $(\zeta, x) \in \partial D \times \partial D_\varepsilon$. Using (4.22), (4.23), (4.25), we obtain by Stokes' formula the equality $\int_{\partial D\times \bar{D}_\varepsilon} \Phi_{1,m} = \int_{\partial D\times\partial D_\varepsilon} incl^*_\varepsilon(\Phi_{0,m})$. $\square$

5 Proof of Lemma 2.1 and Proposition 3.1

5.1 *Continuation of* (V, B) *and notations.* Let $\tilde{V} \in C^2(\mathbb{R}^n, \mathbb{R})$ be such that $\tilde{V} \equiv V$ on $\bar{D}$ and $\|\tilde{V}\|_{C^2,\mathbb{R}^n} < \infty$. Let $\tilde{B} \in C^1(\mathbb{R}^n, A_n(\mathbb{R}))$ (where $A_n(\mathbb{R})$ denotes the space of real antisymmetric matrices) such that $\tilde{B} \equiv B$ on $\bar{D}$ and $\|\tilde{B}\|_{C^1,\mathbb{R}^n} < \infty$. Let ψ be the flow for the differential system

$$\dot{x} = \frac{p}{\sqrt{1 + \frac{p^2}{c^2}}}, \tag{5.1}$$
$$\dot{p} = -\nabla\tilde{V}(x) + \frac{1}{c}\tilde{B}(x)\frac{p}{\sqrt{1 + \frac{p^2}{c^2}}},$$

for $x \in \mathbb{R}^n$ and $p \in \mathbb{R}^n$ (it means that a solution of (5.1), $(x(t), p(t))$, $t \in]t_-, t_+[$, which passes through $(x_0, p_0) \in \mathbb{R}^n \times \mathbb{R}^n$ at time $t = 0$, is written as $(x(t), p(t)) = \psi(t, x_0, p_0)$ for $t \in]t_-, t_+[$). For equation (5.1), the energy

$$E = c^2\sqrt{\frac{1 + |p(t)|^2}{c^2}} + \tilde{V}(x(t)) \tag{5.2}$$

is an integral of motion.

Under the conditions on $\tilde{V}$ and $\tilde{B}$, ψ is defined on $\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^n$ and $\psi\in C^1(\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^n,\mathbb{R}^n\times\mathbb{R}^n)$, and a solution $x(t)$, $t\in]t_-,t_+[$, of (1.1) which starts at $q_0\in D$ at time 0 with velocity v is written as $x(t)=\psi_1(t,x,\frac{v}{\sqrt{1-\frac{v^2}{c^2}}})$, $t\in]t_-,t_+[$ (we write $\psi=(\psi_1,\psi_2)$ where $\psi_i=(\psi_i^1,\ldots,\psi_i^n)\in C^1(\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^n,\mathbb{R}^n)$, $i=1,2$).

For $v\in\mathbb{R}^n$ and $x\in\mathbb{R}^n$, we define the vector $F(x,v)$ of $\mathbb{R}^n$ by

$$F(x,v)=-\nabla\tilde{V}(x)+\frac{1}{c}\tilde{B}(x)v. \tag{5.3}$$

For $x\in\mathbb{R}^n$ and $E>c^2+\tilde{V}(x)$, we denote by $r_{\tilde{V},E}(x)$ the positive real number

$$r_{\tilde{V},E}(x)=c\sqrt{\left(\frac{E-\tilde{V}(x)}{c^2}\right)^2-1}, \tag{5.4}$$

and we denote by $\mathbb{S}^{n-1}_{x,E}$ the following sphere of $\mathbb{R}^n$ of center 0

$$\mathbb{S}^{n-1}_{x,E}=\{p\in\mathbb{R}^n||p|=r_{\tilde{V},E}(x)\}. \tag{5.5}$$

5.2 *Growth estimates for a function g.* Consider the function $g:\mathbb{R}^n\to B_c$ defined by

$$g(x)=\frac{x}{\sqrt{1+\frac{|x|^2}{c^2}}} \tag{5.6}$$

where $x\in\mathbb{R}^n$. This function was considered for example in [J1].

We remind that g has the following simple properties:

$$\begin{aligned}
|\nabla g_i(x)|^2 &\le \frac{1}{1+\frac{|x|^2}{c^2}}, && (5.7)\\
|g(x)-g(y)| &\le \sqrt{n}\sup_{\varepsilon\in[0,1]}\frac{1}{\sqrt{1+\frac{|\varepsilon x+(1-\varepsilon)y|^2}{c^2}}}|x-y|, && (5.8)\\
|\nabla g_i(x)-\nabla g_i(y)| &\le \frac{3\sqrt{n}}{c}\sup_{\varepsilon\in[0,1]}\frac{1}{1+\frac{|\varepsilon x+(1-\varepsilon)y|^2}{c^2}}|x-y|, && (5.9)
\end{aligned}$$

for $x,\ y\in\mathbb{R}^n,\ i=1\ldots n$, and $g=(g_1,\ldots,g_n)$. The function g is an infinitely smooth diffeomorphism from $\mathbb{R}^n$ onto B_c, and its inverse is given by $g^{-1}(x)=\frac{x}{\sqrt{1-\frac{|x|^2}{c^2}}}$, for $x\in B_c$.

5.3 *Proof of Lemma 2.1.* For $q_0,q\in\bar{D}$, $q_0\neq q$, let $t_{+,q_0,q}=\sup\{t>0\mid\psi_1(t,q_0,\bar{k}_{0,V,B}(E,q_0,q))\in D\}$. From Properties (2.1) and (2.2), it follows that $k_{0,V,B}(E,.,.)$ is continuous on $(\bar{D}\times\bar{D})\backslash\bar{G}$ and for $q_0,q\in\bar{D}$,

$q_0 \neq q$, and any $s_1, s_2 \in [0, t_{+,q_0,q}[$, $s_1 \neq s_2$, $\psi_1(s_1, q_0, \bar{k}_{0,V,B}(E, q_0, q)) \neq \psi_1(s_2, q_0, \bar{k}_{0,V,B}(E, q_0, q))$ and $N(q_+) \circ \frac{\partial}{\partial t}\psi_1(t, q_0, \bar{k}_{0,V,B}(E, q_0, q))_{|t=t_{+,q_0,q}}$ is positive, where $q_+ = \psi_1(t_{+,q_0,q}, q_0, \bar{k}_{0,V,B}(E, q_0, q))$ (and where $\circ$ denotes the usual scalar product on $\mathbb{R}^n$). Using also continuity of ψ_1, one obtains that $s_{V,B}(E, q_0, q)$ is continuous on $(\bar{D} \times \bar{D})\backslash\bar{G}$. Then we obtain that $s_{V,B}(E, q_0, q) \in C^1((\bar{D} \times \bar{D})\backslash\bar{G}, \mathbb{R})$ by applying implicit function theorem on maps $m_i : \mathbb{R} \times ((\mathbb{R}^n \times \mathbb{R}^n)\backslash\Delta)$, $(t, x, y) \to y_i - \psi_1^i(t, x, \tilde{k}_{0,V,B}(E, x, y))$, $i = 1 \ldots n$, where $\Delta = \{(x, x) \mid x \in \mathbb{R}^n\}$ and $\tilde{k}_{0,V,B}(E, ., .)$ is a C^1 continuation of $\bar{k}_{0,V,B}(E, ., .)$ on $(\mathbb{R}^n \times \mathbb{R}^n)\backslash\Delta$ (such a continuation exists thanks to (2.2), and note that for any $q_0, q \in \bar{D}$, $q \neq q_0$, $\bar{k}_{V,B}(E, q_0, q) \neq 0$). Note that $k_{V,B}(E, q_0, q) = g(\psi_2(s_{V,B}(E, q_0, q), q_0, \bar{k}_{0,V,B}(E, q_0, q)))$, $q_0, q \in \bar{D}$, $q_0 \neq q$. It remains to prove that $s_{V,B}(E, q_0, q)$ is continuous on $G = \{(q', q') \mid q' \in \bar{D}\}$. Let $q_0 = q \in D$. Let $(q_{0,m})$ and (q_m) be two sequences of points of $\bar{D}$ such that $q_{0,m} \neq q_0$ for all m and $q_{0,m}$ goes to q_0 and q_m goes to $q = q_0$ as $m \to +\infty$. Let $R = \limsup_{m\to+\infty} s_{V,B}(E, q_{0,m}, q_m) \in [0, +\infty]$. We shall prove that $R = 0$. Assume that $R > 0$. Note that by conservation of energy $|\bar{k}_{0,V,B}(E, q_{0,m}, q_m)| \leq c\sqrt{\left(\frac{E+\|V\|_\infty}{c^2}\right)^2 - 1}$. Using definition of R and compactness of the closed ball of $\mathbb{R}^n$ whose radius is $c\sqrt{\left(\frac{E+\|V\|_\infty}{c^2}\right)^2 - 1}$ and whose centre is 0, we obtain that there exist subsequences of $q_{0,m}$ and q_m (respectively still denoted by $q_{0,m}$ and q_m) such that

$$\lim_{m\to+\infty} s_{V,B}(E, q_{0,m}, q_m) = R, \tag{5.10}$$

$$\bar{k}_{0,V,B}(E, q_{0,m}, q_m) \text{ converges to some } k \in \mathbb{R}^n. \tag{5.11}$$

Using conservation of energy, we obtain that

$$|k| = c\sqrt{\left(\frac{E - V(q_0)}{c^2}\right)^2 - 1}. \tag{5.12}$$

Using (5.11) and (5.10) and continuity of ψ_1, we obtain that

$$\psi_1(t, q_0, k) = \lim_{m\to+\infty} \psi_1(t, q_{0,m}, \bar{k}_{0,V,B}(E, q_{0,m}, q_m)), \text{ for all } t \in [0, R[. \tag{5.13}$$

For all m and $t \in [0, s_{V,B}(E, q_{0,m}, q_m)[$, $\psi_1(t, q_{0,m}, \bar{k}_{0,V,B}(E, q_{0,m}, q_m)) \in \bar{D}$. Hence using (5.13), we obtain that

$$\psi_1(t, q_0, k) \in \bar{D},\ t \in [0, R[. \tag{5.14}$$

In addition,

$$\psi_1(0, q_0, k) = q_0 \in D. \tag{5.15}$$

Then $R \neq +\infty$ (otherwise this would contradict (2.1), in particular the fact that the solution of (1.1) under condition (1.3) with energy E, which starts at time 0 at $q_0 = \psi_1(0, q_0, k)$, reaches the boundary ∂D at a time $t_+ > 0$ and satisfies the estimate $\frac{\partial \psi_1}{\partial t}(t_+, q_0, k) \circ N(\psi_1(t_+, q_0, k)) > 0)$.

Using continuity of ψ_1 and $\lim_{m\to+\infty} q_{0,m} = q_0$, $\lim_{m\to+\infty} q_m = q_0$, $\lim_{m\to+\infty} s_{V,B}(E, q_{0,m}, q_m) = R$, $\lim_{m\to+\infty} \bar{k}_{0,V,B}(E, q_{0,m}, q_m) = k$ and the definition of $s_{V,B}(E, q_{0,m}, q_m)$, we obtain that

$$\begin{aligned} \psi_1(R, q_0, k) &= \lim_{m\to+\infty} \psi_1(s_{V,B}(E, q_{0,m}, q_m), q_0, \bar{k}_{0,V,B}(E, q_{0,m}, q_m)) \\ &= \lim_{m\to+\infty} q_m = q_0. \end{aligned} \tag{5.16}$$

Properties (5.16), (5.15), (5.14) and (2.1) imply $R = 0$, which contradicts the assumption $R > 0$. Finally we proved that $s_{V,B}(E, ., .) \in C((\bar{D} \times \bar{D})\backslash\partial G, \mathbb{R})$.

Let $x_0 \in D$. From (2.3), it follows that for sufficiently small positive ε, E is greater than $E(\|\tilde{V}\|_{C^2,D_{x_0,\varepsilon}}, \|\tilde{B}\|_{C^1,D_{x_0,\varepsilon}}, D_{x_0,\varepsilon})$ where $D_{x_0,\varepsilon} = \{x_0 + (1+\varepsilon)(x - x_0) \mid x \in D\}$. Hence one obtains that solutions of energy E for equation (5.1) in $D_{x_0,\varepsilon}$ also have properties (2.1) and (2.2); and replacing V, B, and D by $\tilde{V}$, $\tilde{B}$ and $D_{x_0,\varepsilon}$ above in the proof, one obtains that $s_{\tilde{V},\tilde{B}}(E, ., .)$ is continuous on $(\bar{D}_{x_0,\varepsilon} \times \bar{D}_{x_0,\varepsilon})\backslash\{(q,q) \mid q \in \partial D_{x_0,\varepsilon}\}$ ($s_{\tilde{V},\tilde{B}}(E, q_0', q')$, $(q_0', q') \in \bar{D}_{x_0,\varepsilon} \times \bar{D}_{x_0,\varepsilon}$, are defined as $s_{V,B}(E, q_0, q)$, $(q_0, q) \in \bar{D} \times \bar{D}$, are defined in Subsection 2.1). Now, using also $\bar{D} \subseteq D_{x_0,\varepsilon}$ and the equality $s_{V,B}(E, q_0, q) = s_{\tilde{V},\tilde{B}}(E, q_0, q)$ for q_0, $q \in \bar{D}$, one obtains $s_{V,B}(E, ., .) \in C(\bar{D} \times \bar{D}, \mathbb{R})$ (the equality $s_{V,B}(E, q_0, q) = s_{\tilde{V},\tilde{B}}(E, q_0, q)$ for q_0, $q \in \bar{D}$, follows from the fact that if $(x(t), p(t))$ is solution of (5.1) in D, then $(x(t), p(t))$ is also solution of (5.1) in $D_{x_0,\varepsilon}$).

Lemma 2.1 is proved. $\square$

5.4 *Proof of Proposition 3.1.* From Lemma 2.1, $\psi \in C^1(\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^n, \mathbb{R}^n\times\mathbb{R}^n)$, $\mathbf{A} \in C^1(\bar{D}, \mathbb{R}^n)$, it follows that $\mathcal{S}_{0_{V,\mathbf{A},E}} \in C(\bar{D}\times\bar{D}, \mathbb{R})$ and $\mathcal{S}_{0_{V,\mathbf{A},E}} \in C^1((\bar{D}\times\bar{D})\backslash\bar{G}, \mathbb{R})$. Equalities (3.9) and (3.10) are known equalities (see Section 46 and further Sections of [A]). Statements (3.8), (3.11), (3.12) follow from (3.9) and (3.10). We shall prove (3.13). We omit indices $_{V,B}$ for $s_{V,B}$, $\bar{k}_{0,V,B}$ and $\bar{k}_{V,B}$ where $\bar{k}_0$, $\bar{k}$ are defined by (3.1). Using the equality $y - x = \int_0^{s(E,x,y)} \frac{\partial\psi_1}{\partial t}(t, x, \bar{k}_0(E, x, y))dt$ and estimate $|\frac{\partial\psi_1}{\partial t}(t, x, \bar{k}_0(E, x, y))| \leq c$, we obtain

$$|y - x| \leq cs(E, x, y), \text{ for all } x, y \in \bar{D},\ y \neq x. \tag{5.17}$$

Derivating equality $\psi_1(s(E, x, y), x, \bar{k}_0(E, x, y)) = y$ with respect to y_i, we obtain that

$$e_i = \Big(\frac{\partial\psi_1^j}{\partial\bar{k}}(s(E, x, y), x, \bar{k}_0(E, x, y)) \circ \frac{\partial\bar{k}_0}{\partial y_i}(E, x, y)\Big)_{j=1..n} \tag{5.18}$$

$$+\frac{\partial s}{\partial y_i}(E,x,y)\frac{\partial \psi_1}{\partial s}(s(E,x,y),x,\bar{k}_0(E,x,y)),$$

for any $x,y\in\bar{D}$, $x\neq y$ and where $(e_1,\ldots,e_n)$ is the canonical basis of $\mathbb{R}^n$ (and where $\circ$ denotes the usual scalar product on $\mathbb{R}^n$). For $t\in\mathbb{R}$, $x\in\bar{D}$, and $k\in\mathbb{R}^n$ and $j=1..n$ the following equality is valid: $\psi_1^j(t,x,k)=x_j+tg_j(k)+\int_0^t\left[g_j(k+\int_0^s F(\psi_1(\sigma,x,k),g(\psi_2(\sigma,x,k)))d\sigma)-g_j(k)\right]ds$. Hence we obtain that for $t\in\mathbb{R}$, $x\in\bar{D}$, $k=(k_1,\ldots,k_n)\in\mathbb{R}^n$, and $j=1..n$,

$$\frac{\partial \psi_1^j}{\partial k_l}(t,x,k)=t\frac{\partial g_j}{\partial k_l}(k)+\int_0^t\left[\frac{\partial g_j}{\partial k_l}(k+\int_0^s F(\psi_1(\sigma,x,k),g(\psi_2(\sigma,x,k)))\right.$$
$$\left.d\sigma)-\frac{\partial g_j}{\partial k_l}(k)\right]ds+\int_0^t\nabla g_j(k+\int_0^s F(\psi_1(\sigma,x,k),g(\psi_2(\sigma,x,k)))d\sigma)\circ$$
$$\int_0^s\frac{\partial}{\partial k_l}F(\psi_1,g(\psi_2))_{|(\sigma,x,k)}d\sigma ds, \tag{5.19}$$

for any $l=1..n$. Define

$$R=\sup_{(x',y')\in\bar{D}} s(E,x',y'),$$
$$M_3=\sup_{\substack{t\in[0,R],x'\in\bar{D},l=1\ldots n\\ |k|\le c\sqrt{\left(\frac{\sup_{x'\in\bar{D}}E-V(x')}{c^2}\right)^2-1}}}\left|\frac{\partial}{\partial k_l}F(\psi_1,g(\psi_2))_{|(t,x',k)}\right|,$$
$$M_4=\max(M_3,\sqrt{n}\|V\|_{C^2,D}+n\|B\|_{C^1,D}).$$

Then using (5.19) and growth properties of g, we obtain that

$$\left|\frac{\partial \psi_1^j}{\partial k_l}(s(E,x,y),x,\bar{k}_0(E,x,y))-s(E,x,y)\frac{\partial g_j}{\partial k_l}(\bar{k}_0(E,x,y))\right|$$
$$\le M_4 s(E,x,y)^2(1+\frac{3\sqrt{n}}{c}), \tag{5.20}$$

for $x,y\in\bar{D}$, $x\neq y$ and $j,l=1..n$. where $\bar{k}_0$ is defined by (3.1).

Let $x,y\in\bar{D}$, $x\neq y$. Using the identity

$$\bar{k}(E,x,y)=\bar{k}_0(E,x,y)+\int_0^{s(E,x,y)}\Big(-\nabla V(\psi_1(s,x,\bar{k}_0(E,x,y)))$$
$$+\frac{1}{c}B(\psi_1(s,x,\bar{k}_0(E,x,y)))g(\psi_2(s,x,\bar{k}_0(E,x,y)))\Big)ds,$$

we obtain the following estimate

$$|\bar{k}(E,x,y)-\bar{k}_0(E,x,y)|\le M_5 s(E,x,y), \tag{5.21}$$

where $M_5 = \sqrt{n}\|V\|_{C^2,D} + n\|B\|_{C^1,D}$. Using (3.2), we obtain that

$$\bar{k}_0(E,x,y) \circ \frac{\partial \bar{k}_0}{\partial y_i}(E,x,y) = \frac{1}{2}\frac{\partial |\bar{k}_0|^2}{\partial y_i}(E,x,y) = 0, \tag{5.22}$$

for $i = 1..n$. From (3.2), (5.21) and (5.22), it follows that

$$\begin{aligned}
&|\frac{\bar{k}(E,x,y)}{|\bar{k}(E,x,y)|} \circ \frac{\partial \bar{k}_0}{\partial y_i}(E,x,y)| \leq \frac{1}{|\bar{k}(E,x,y)|}|(\bar{k}(E,x,y) - \bar{k}_0(E,\\
&x,y)) \circ \frac{\partial \bar{k}_0}{\partial y_i}(E,x,y)| + \frac{1}{|\bar{k}(E,x,y)|}|\bar{k}_0(E,x,y) \circ \frac{\partial \bar{k}_0}{\partial y_i}(E,x,y)|\\
&\leq c^{-1}\left(\left(\frac{\inf_{x'\in\bar{D}} E - V(x')}{c^2}\right)^2 - 1\right)^{-\frac{1}{2}} M_5 s(E,x,y)|\frac{\partial \bar{k}_0}{\partial y_i}(E,x,y)|,
\end{aligned} \tag{5.23}$$

for $i = 1..n$.

Let $(v^1,\ldots,v^{n-1})$ be an orthonormal family of $\mathbb{R}^n$ such that $(\frac{\bar{k}(E,x,y)}{|\bar{k}(E,x,y)|}, v^1, \ldots, v^{n-1})$ is an orthonormal basis of $\mathbb{R}^n$. Note that using definition of g and (5.22), we obtain

$$\begin{aligned}
&(\nabla g_j(\bar{k}_0(E,x,y)) \circ \frac{\partial \bar{k}_0}{\partial y_i}(E,x,y))_{j=1..n} =\\
&\left(1 + \frac{|\bar{k}_0(E,x,y)|^2}{c^2}\right)^{-1/2} \frac{\partial \bar{k}_0}{\partial y_i}(E,x,y),\ i = 1..n.
\end{aligned} \tag{5.24}$$

Hence using (5.24), (5.20) and (5.18) (and $k(E,x,y) \circ v^h = 0$), we obtain

$$\begin{aligned}
&s(E,x,y)|\frac{\partial \bar{k}_0}{\partial y_i}(E,x,y) \circ v^h| = \sqrt{1 + \frac{|\bar{k}_0(E,x,y)|^2}{c^2}} s(E,x,y)\\
&\times|(\nabla g_j(\bar{k}_0(E,x,y)) \circ \frac{\partial \bar{k}_0}{\partial y_i}(E,x,y))_{j=1..n} \circ v^h|\\
\leq\ & \sqrt{1 + \frac{|\bar{k}_0(E,x,y)|^2}{c^2}} \Bigg[nM_4 s(E,x,y)^2(1 + \frac{3}{c})|\frac{\partial \bar{k}_0}{\partial y_i}(E,x,y)|\\
&+ |(\frac{\partial \psi_1^j}{\partial k}(s(E,x,y), x, \bar{k}_0(E,x,y)) \circ \frac{\partial \bar{k}_0}{\partial y_i}(E,x,y))_{j=1..n} \circ v^h|\Bigg]\\
=\ & \sqrt{1 + \frac{|\bar{k}_0(E,x,y)|^2}{c^2}} \left[nM_4 s(E,x,y)^2(1 + \frac{3}{c})|\frac{\partial \bar{k}_0}{\partial y_i}(E,x,y)| + |v_i^h|\right]\\
\leq\ & \frac{\sup_{x'\in\bar{D}} E - V(x')}{c^2}(nM_4(1 + \frac{3}{c})s(E,x,y)^2|\frac{\partial \bar{k}_0}{\partial y_i}(E,x,y)| + 1),
\end{aligned} \tag{5.25}$$

for $i = 1 \dots n$. Let $M_6 = c^{-1}\left((c^{-2}\inf_{x'\in\bar D}(E - V(x')))^2 - 1\right)^{-\frac{1}{2}} M_5 + c^{-2}\times$ $\sup_{x'\in\bar D}(E - V(x'))(nM_4(1+\frac{3}{c})+1)$. From (5.23) and (5.25), it follows that

$$s(E,x,y)|\frac{\partial \bar k_0}{\partial y_i}(E,x,y)| \le \sqrt{n}M_6(1 + s(E,x,y)(s(E,x,y)|\frac{\partial \bar k_0}{\partial y_i}(E,x,y)|)), \tag{5.26}$$

for $i = 1 \dots n$.

Using uniform continuity of $s(E,.,.)$ on $\bar D \times \bar D$, we obtain that there exists some $\eta > 0$ such that if $x, y \in \bar D$, $|x-y| < \eta$, then $\sqrt{n}M_6 s(E,x,y) \le \frac{1}{2}$. Then, using (5.26), we obtain that $s(E,x,y)|\frac{\partial \bar k_0}{\partial y_i}(E,x,y)| \le 2\sqrt{n}M_6$, for $x, y \in \bar D$, $|x-y| < \eta$ and $i = 1..n$. Now using the continuous differentiability of $\bar k_0(E,.,.)$ on $(\bar D\times\bar D)\backslash \bar G$, we obtain that $|\frac{\partial \bar k_0}{\partial y_i}(E,x,y)| \le \frac{M_i'}{s(E,x,y)}$ for $x, y \in \bar D$, $x \ne y$ and where $M_i' = \max(2M_6\sqrt{n}, R\sup_{x',y'\in\bar D, |x'-y'|\ge\eta}|\frac{\partial \bar k_0}{\partial y_i}(E,x,y)|)$. Putting $M_2 = \sup_{i=1..n} cM_i'$ and using (5.17) and (3.11), we obtain (3.13). $\square$

6 Proof of Properties (2.1) and (2.2)

In this Section we first consider solutions $x(t)$ of (5.1) in an open bounded subset Ω of $\mathbb{R}^n$ (see Subsection 6.1) and we give properties of these solutions at fixed and sufficiently large energy (see Subsections 6.3 and 6.4) (Ω should be thought as an open neighborhood of D). Using these properties we prove Properties 2.1 and 2.2 (see Subsection 6.5). Subsections 6.6, 6.7, 6.8, 6.9 are devoted to the proof of Propositions 6.1, 6.2, 6.3, 6.4 formulated in Subsection 6.4.

We keep notations of Subsections 5.1, 5.2.

6.1 *Additional notations.* Let Ω be a bounded open subset of $\mathbb{R}^n$ with frontier $\partial\Omega$. We define a positive number $\delta(\Omega)$ by

$$\delta(\Omega) = \sup_{x\in\Omega}|x|. \tag{6.1}$$

We consider the following equation in Ω :

$$\dot p = F(x,\dot x),\ p = \frac{\dot x}{\sqrt{1-\frac{|\dot x|^2}{c^2}}},\ x\in\Omega,\ p\in\mathbb{R}^n. \tag{6.2}$$

where the force $F(x,\dot x) = -\nabla \tilde V(x) + \frac{1}{c}\tilde B(x)\dot x$ is defined by (5.3). For the equation (6.2), the energy $E = c^2\sqrt{\frac{1+|p(t)|^2}{c^2}} + \tilde V(x(t))$ is an integral of motion.

Note that if $x(t)$, $t \in]t_-, t_+[$, is a solution of (6.2) which starts at $x_0 \in \Omega$ at time 0 with velocity v then $x(t) = \psi_1(t, x_0, g^{-1}(v))$, $t \in]t_-, t_+[$, where the function g is defined by (5.6) and where $\psi = (\psi_1, \psi_2)$ is the flow of the differential system (5.1). We obtain, in particular, $x(t) \to \psi_1(t_\pm, x_0, g^{-1}(v))$, as $t \to t_\pm$, and $\dot{x}(t) \to g(\psi_2(t_\pm, x_0, g^{-1}(v)))$, as $t \to t_\pm$.

We denote by Λ the open subset of $\mathbb{R} \times \Omega \times \mathbb{R}^n$ where the flow of the differential system (6.2) is defined, i.e.

$$\Lambda = \{(t, x, p) \in \mathbb{R} \times \Omega \times \mathbb{R}^n \mid \forall s \in [0, t]\ \psi_1(s, x, p) \in \Omega\},$$

where $\psi = (\psi_1, \psi_2)$ is the flow of the differential system (5.1).

For $E > c^2 + \sup_{x\in\Omega} \tilde{V}(x)$, we denote by $\mathcal{V}_E$ the following smooth $2n-1$-dimensional submanifold of $\mathbb{R}^{2n}$

$$\mathcal{V}_E = \{(x, p) \in \Omega \times \mathbb{R}^n \mid |p| = r_{\tilde{V},E}(x)\}, \tag{6.3}$$

where $r_{\tilde{V},E}(x)$ is defined by (5.4).

For $E > c^2 + \sup_{x\in\Omega} \tilde{V}(x)$, we also consider the map $\varphi_E \in C^1(\Lambda \cap (]0, +\infty[\times \mathcal{V}_E), \Omega \times \Omega)$, defined by

$$\varphi_E(t, x, p) = (x, \psi_1(t, x, p)), \quad \text{for } (t, x, p) \in \Lambda \cap (]0, +\infty[\times \mathcal{V}_E). \tag{6.4}$$

6.2 *Estimates for the force F.* We define the nonnegative real number $\beta(\tilde{V}, \tilde{B}, \Omega)$ by

$$\beta(\tilde{V}, \tilde{B}, \Omega) = \max\left(\sup_{\substack{x\in\Omega \\ \alpha\in(\mathbb{N}\cup\{0\})^n \\ |\alpha|\le 2}} |\partial_x^\alpha \tilde{V}(x)|, \sup_{\substack{x\in\Omega \\ \alpha'\in(\mathbb{N}\cup\{0\})^n \\ |\alpha'|\le 1}} |\partial_x^{\alpha'} \tilde{B}_{i,j}(x)|\right). \tag{6.5}$$

The following estimates are valid:

$$|F(x, v)| \le n\beta(\tilde{V}, \tilde{B}, \Omega)(\frac{1}{c}|v| + 1), \tag{6.6}$$

$$|F(x, v) - F(x', v')| \le n\beta(\tilde{V}, \tilde{B}, \Omega)\left[|x - x'|\left(1 + \frac{|v'|}{c}\right) + \frac{1}{c}|v' - v|\right], \tag{6.7}$$

for $x, x' \in \Omega$ and $v, v' \in \mathbb{R}^n$.

6.3 *Some constants.* For $x \in \Omega$ and $E > c^2 + \sup_{x' \in \Omega} V(x')$, we define the following real constants

$$C_1 = 2c^2 + \sup_{x' \in \Omega} \left(\tilde{V}(x') + 8|x'| \left(|\nabla \tilde{V}(x')| + \sum_{i,j=1\dots n} |\tilde{B}_{i,j}(x')| \right) \right), \tag{6.8}$$

$$C_2 = c^2 \sqrt{1 + \frac{800 n^2 \beta(\tilde{V}, \tilde{B}, \Omega)^2 \delta(\Omega)^2}{c^4}} + \sup_{x' \in \Omega} \tilde{V}(x'), \tag{6.9}$$

$$C_3 = C_4 \left(1 + 5\delta(\Omega) C_5 e^{5\delta(\Omega) C_5}\right) \tag{6.10}$$

$$\begin{aligned} C_4 = & \left(1 + \frac{10\sqrt{2} n^{3/2} \delta(\Omega) \beta(\tilde{V}, \tilde{B}, \Omega)}{E - \tilde{V}(x)} e^{\frac{10\sqrt{2}\delta(\Omega) n^{3/2} \beta(\tilde{V}, \tilde{B}, \Omega)}{E - \tilde{V}(x)}} \right) \\ & \times \frac{10 n^{3/2} \delta(\Omega)^2 \beta(\tilde{V}, \tilde{B}, \Omega)}{E - \tilde{V}(x)} \left(5 + \frac{600 n^{3/2} \delta(\Omega) \beta(\tilde{V}, \tilde{B}, \Omega)}{E - \tilde{V}(x)} + 24 n^{1/2}\right) \\ & + \frac{20\sqrt{2} n^2 \delta(\Omega) \beta(\tilde{V}, \tilde{B}, \Omega)}{E - \tilde{V}(x)} e^{\frac{10\sqrt{2}\delta(\Omega) n^{3/2} \beta(\tilde{V}, \tilde{B}, \Omega)}{E - \tilde{V}(x)}} + \frac{40 n^{3/2} \delta(\Omega) \beta(\tilde{V}, \tilde{B}, \Omega)}{c^2 \sqrt{\left(\frac{E - \tilde{V}(x)}{c^2}\right)^2 - 1}}, \end{aligned} \tag{6.11}$$

$$\begin{aligned} C_5 = & \left(1 + \frac{10\sqrt{2} n^{3/2} \delta(\Omega) \beta(\tilde{V}, \tilde{B}, \Omega)}{E - \tilde{V}(x)} e^{\frac{10\sqrt{2}\delta(\Omega) n^{3/2} \beta(\tilde{V}, \tilde{B}, \Omega)}{E - \tilde{V}(x)}} \right) \\ & \times \frac{20 n^{3/2} \beta(\tilde{V}, \tilde{B}, \Omega) \delta(\Omega)}{E - \tilde{V}(x)} \left(1 + \frac{120 n^{3/2} \beta(\tilde{V}, \tilde{B}, \Omega) \delta(\Omega)}{E - \tilde{V}(x)}\right), \end{aligned} \tag{6.12}$$

$$C_6 = \inf_{x' \in \Omega} \left(\sqrt{1 - \left(\frac{c^2}{E - \tilde{V}(x')}\right)^2} - \frac{20 n^2 \beta(\tilde{V}, \tilde{B}, \Omega) \delta(\Omega)}{E - \tilde{V}(x')} \right), \tag{6.13}$$

$$\begin{aligned} C_7 = \inf_{x' \in \Omega} & \left(c\sqrt{1 - \left(\frac{c^2}{E - \tilde{V}(x')}\right)^2} - \frac{5c(n+1)^{1/2} n^2 \delta(\Omega)}{E - \tilde{V}(x')} \right. \\ & \times \beta(\tilde{V}, \tilde{B}, \Omega) e^{\frac{10 n^{3/2} \beta(\tilde{V}, \tilde{B}, \Omega) \delta(\Omega)}{E - \tilde{V}(x')} \left(1 + 2cn^{1/2} e^{\frac{10 n^{3/2} \beta(\tilde{V}, \tilde{B}, \Omega) \delta(\Omega)}{E - \tilde{V}(x')}}\right)} \\ & \left. \times \frac{c r_{\tilde{V},E}(x')}{E - \tilde{V}(x')} \left[12\sqrt{n} + 1 + 10\sqrt{n}\delta(\Omega)\right] \right). \end{aligned} \tag{6.14}$$

Now assume that Ω is a bounded strictly convex (in the strong sense) open domain of $\mathbb{R}^n$ with C^2 boundary. Let χ_Ω be a C^2 defining function for Ω, i.e. $\Omega = \chi_\Omega^{-1}(]-\infty, 0[)$ and $\partial\Omega = \chi_\Omega^{-1}(\{0\})$ and for all $x \in \partial\Omega$ $\nabla \chi_\Omega(x) \neq 0$ and the Hessian matrix $Hess\chi_\Omega(x)$ of χ_Ω at x is a symmetric positive definite matrix. For $E > c^2 + \sup_{x\in\Omega}\tilde V(x)$, we define the real constant $C_8(E,\tilde V,\tilde B,\Omega)$ by

$$C_8 = C_{10}(\Omega)\left(1-\left(\frac{c^2}{E-\sup_{y\in\Omega}\tilde V(y)}\right)^2\right) - \frac{4nC_9(\Omega)\beta(\tilde V,\tilde B,\Omega)}{E-\sup_{y\in\Omega}\tilde V(y)}, \quad (6.15)$$

where $C_9(\Omega)$ and $C_{10}(\Omega)$ are the two positive real numbers defined by

$$\begin{aligned} C_9(\Omega) &= \sup_{x\in\partial\Omega}|\nabla\chi_\Omega(x)|, && (6.16)\\ C_{10}(\Omega) &= \inf_{\substack{x\in\partial\Omega\\ v\in\mathbb{S}^{n-1}}}|Hess\chi_\Omega(x)(v,v)|. && (6.17)\end{aligned}$$

Note that from (6.10)-(6.15), it follows that

$$\begin{array}{l} \sup_{x\in\Omega}C_3(E,x,\tilde V,\tilde B,\Omega)\to 0,\ \text{as } E\to+\infty,\\ C_6(E,\tilde V,\tilde B,\Omega)\to 1>0,\ \text{as } E\to+\infty,\\ C_7(E,\tilde V,\tilde B,\Omega)\to c>0,\ \text{as } E\to+\infty,\\ C_8(E,\tilde V,\tilde B,\Omega)\to C_{10}(\Omega)>0,\ \text{as } E\to+\infty. \end{array} \quad (6.18)$$

When Ω is strictly convex in the strong sense with C^2 boundary, then one can relate an upper bound for the real constant $E(\|\tilde V\|_{C^2,\Omega},\|\tilde B\|_{C^1,\Omega},\Omega)$ (mentioned in Subsection 2.1) with constants C_1, C_2, $\sup_{x\in\Omega}C_3$, C_6, C_7 and C_8 (see Subsections 6.4 and 6.5).

Remark 6.1. We remind that $\tilde V$ is a C^1 continuation of V on $\mathbb{R}^n$ and that $\tilde B\in C^1(\mathbb{R}^n,A_n(\mathbb{R}))$ is such that $\tilde B\equiv B$ on $\bar D$. Note that from (6.8)-(6.15) it follows that $C_1(\tilde V,\tilde B,D)$, $C_2(\tilde V,\tilde B,D)$, $\sup_{x\in D}C_3(E,x,\tilde V,\tilde B,D)$, $C_6(E,\tilde V,\tilde B,D)$, $C_7(E,\tilde V,\tilde B,D)$ and $C_8(E,\tilde V,\tilde B,D)$ depend only on (V,B) and D.

6.4 *Properties of the first component of the flow of* (6.2) *at fixed and sufficiently large energy E.* The following Proposition 6.1 gives an upper bound for living time for solutions of (6.2) with energy E when E is sufficiently large.

Proposition 6.1. *Let*

$$E\geq C_1(\tilde V,\tilde B,\Omega), \quad (6.19)$$

where C_1 is defined by (6.8)*. Let $x:]t_-,t_+[\to\Omega$ be a solution of* (6.2) *with energy E, where $t_\pm\in\mathbb{R}\cup\{\pm\infty\}$. Then the following statement holds: t_-,t_+ are finite and they satisfy the following estimate*

$$|t_+-t_-|\leq\frac{5\delta(\Omega)}{c}, \quad (6.20)$$

where $\delta(\Omega)$ *is defined by* (6.1)*.*

A proof of Proposition 6.1 is given in Subsection 6.6.

For $E \geq C_1(\tilde{V}, \tilde{B}, \Omega)$ (C_1 is defined by (6.8)) and for $(x,p) \in \mathcal{V}_E$, we define the real numbers $t_{+,x,p}$ and $t_{-,x,p}$ by

$$t_{+,x,p} = \sup\{t > 0 \mid (t,x,p) \in \Lambda\}, \tag{6.21}$$
$$t_{-,x,p} = \inf\{t < 0 \mid (t,x,p) \in \Lambda\}. \tag{6.22}$$

The following Proposition 6.2 gives, in particular, a one-to-one property of the map φ_E defined by (6.4).

Proposition 6.2. *Let*

$$E \geq \max(C_1(\tilde{V}, \tilde{B}, \Omega), C_2(\tilde{V}, \tilde{B}, \Omega)), \tag{6.23}$$

where constants C_1 *and* C_2 *are defined by* (6.8) *and* (6.9)*. Let* $x \in \Omega$ *and let* $p_1, p_2 \in \mathbb{S}^{n-1}_{x,E}$ *(defined by* (5.5)*). Then the following estimate is valid:*

$$\big||\psi_1(t_1,x,p_1) - \psi_1(t_2,x,p_2)| - |t_1v_1 - t_2v_2|\big| \leq C_3|t_1v_1 - t_2v_2|, \tag{6.24}$$

for $(t_1,t_2) \in [0, t_{+,x,p_1}[\times[0, t_{+,x,p_2}[$*, where* $v_i = \frac{p_i}{\sqrt{1-\frac{p_i^2}{c^2}}}$*,* $i = 1,2$*, and where* $C_3 = C_3(E,x,\tilde{V},\tilde{B},\Omega)$ *is defined by* (6.10)*.*

A proof of Proposition 6.2 is given in Subsection 6.7. We remind that

$$\sup_{x\in\Omega} C_3(E,x,\tilde{V},\tilde{B},\Omega) \to 0, \text{ as } E \to +\infty \text{ (see (6.18))}. \tag{6.25}$$

Taking account of (6.25) and the equality $\psi_1(0,x,p) = x$ for any $(x,p) \in \mathcal{V}_E$ and taking account of Proposition 6.2, we obtain that at fixed and sufficiently large energy E the map φ_E defined by (6.4) is one-to-one and its range is included in $(\Omega \times \Omega)\backslash\{(x,x) \mid x \in \Omega\}$.

The following Proposition 6.3 is proved in Subsection 6.8.

Proposition 6.3. *Assume that*

$$\begin{array}{l} E \geq C_1(\tilde{V}, \tilde{B}, \Omega), \\ E \geq c^2\sqrt{1 + \frac{400n^2\beta(\tilde{V},\tilde{B},\Omega)^2\delta(\Omega)^2}{c^4}} + \sup_{x\in\Omega}\tilde{V}(x), \\ \min(C_6(E,\tilde{V},\tilde{B},\Omega), C_7(E,\tilde{V},\tilde{B},\Omega)) > 0, \end{array} \tag{6.26}$$

where C_1*,* C_6 *and* C_7 *are defined by* (6.8)*,* (6.13) *and* (6.14)*. Then the map* φ_E *defined by* (6.4) *is a local* C^1 *diffeomorphism at any point* $(t,x,p) \in \Lambda \cap (]0,+\infty[\times\mathcal{V}_E)$ *.*

Now assume that Ω is a bounded strictly convex (in the strong sense) open domain of $\mathbb{R}^n$ with C^2 boundary. Let χ_Ω be a C^2 defining function for Ω. For $E > c^2 + \sup_{x\in\Omega} \tilde V(x)$, real constant $C_8(E, \tilde V, \tilde B, \Omega)$ is defined by (6.15) with respect to χ_Ω.

The following Proposition 6.4 gives a surjectivity property of the map φ_E defined by (6.4) at fixed and sufficiently large energy E.

Proposition 6.4. *Let*

$$\begin{aligned}
&E \geq C_1(\tilde V, \tilde B, \Omega),\\
&E \geq c^2\sqrt{1 + \tfrac{400n^2\beta(\tilde V,\tilde B,\Omega)^2\delta(\Omega)^2}{c^4}} + \sup_{x\in\Omega} \tilde V(x),\\
&\min(C_6(E, \tilde V, \tilde B, \Omega), C_7(E, \tilde V, \tilde B, \Omega), C_8(E, \tilde V, \tilde B, \Omega)) > 0.
\end{aligned} \tag{6.27}$$

where C_1, C_6, C_7 *and* C_8 *are defined by* (6.8), (6.13), (6.14) *and* (6.15).

Then $(\Omega \times \Omega)\backslash\{(x, x) \mid x \in \Omega\}$ *is included in the range of the map* φ_E *defined by* (6.4).

A proof of Proposition 6.4 is given in Subsection 6.9.

Taking account of Propositions 6.2, 6.3, 6.4, we obtain, in particular, that at fixed and sufficiently large energy E the map φ_E defined by (6.4) is a C^1 diffeomorphism from $\Lambda \cap (]0, +\infty[\times \mathcal{V}_E)$ onto $(\Omega \times \Omega)\backslash\{(x, x) \mid x \in \Omega\}$.

Now we are ready to prove Properties 2.1 and 2.2.

6.5 *Final part of the proof of Properties (2.1) and (2.2).* Let χ_D be a C^2 defining function for D, i.e. $\chi_D \in C^2(\mathbb{R}^n, \mathbb{R})$ and $D = \chi_D^{-1}(] - \infty, 0[)$, and $\partial D = \chi_D^{-1}(\{0\})$, and for all $x \in \partial D$ $\nabla\chi_D(x) \neq 0$ and the Hessian matrix $Hess\chi_D(x)$ of χ_D at x is a symmetric definite positive matrix.

Let $x_0 \in D$. For $\varepsilon > 0$, we define the open neighborhood Ω_ε of $\bar D$ by $\Omega_\varepsilon = \{x_0 + (1 + \varepsilon)(x' - x_0) \mid x' \in D\}$. Then Ω_ε is also a bounded strictly convex in the strong sense open domain of $\mathbb{R}^n$ with C^2 boundary and the map $\chi_{\Omega_\varepsilon} \in C^2(\mathbb{R}^n, \mathbb{R})$ defined by $\chi_{\Omega_\varepsilon}(x) = \chi_D(x_0 + \frac{x-x_0}{1+\varepsilon}),\ x \in \mathbb{R}^n$, is a C^2 defining function for Ω_ε. In addition, note that

$$\begin{aligned}
&x \in \partial\Omega_\varepsilon \Leftrightarrow x_0 + \tfrac{x-x_0}{1+\varepsilon} \in \partial D,\\
&\nabla\chi_{\Omega_\varepsilon}(x) = (1 + \varepsilon)^{-1}\nabla\chi_D(x_0 + \tfrac{x-x_0}{1+\varepsilon}),\ x \in \mathbb{R}^n,\\
&Hess\chi_{\Omega_\varepsilon}(x) = (1 + \varepsilon)^{-2}Hess\chi_D(x_0 + \tfrac{x-x_0}{1+\varepsilon}),\ x \in \mathbb{R}^n,\\
&\sup_{x\in\Omega_\varepsilon}\big(\inf\{|x - y| \mid y \in \bar D\}\big) = \varepsilon \sup\{|x - x_0| \mid x \in D\} \underset{\varepsilon\to 0^+}{\longrightarrow} 0.
\end{aligned} \tag{6.28}$$

Note also that $\Omega_{\varepsilon_2} \subseteq \Omega_{\varepsilon_1}$ if $0 < \varepsilon_2 < \varepsilon_1$.

Let $E > c^2 + \sup_{x\in D} V(x)$. Assume that

$$\begin{aligned}
&E > \max(C_1(\tilde V, \tilde B, D), C_2(\tilde V, \tilde B, D)),\\
&\sup_{x\in\Omega} C_3(E, x, \tilde V, \tilde B, D) < 1,\\
&\min(C_6(E, \tilde V, \tilde B, D), C_7(E, \tilde V, \tilde B, D), C_8(E, \tilde V, \tilde B, D)) > 0,
\end{aligned} \tag{6.29}$$

where C_1, C_2, C_3, C_6, C_7 and C_8 are defined by (6.8), (6.9), (6.10), (6.13), (6.14) and (6.15) (taking account of (6.18), we obtain that if E is sufficiently large, then (6.29) is satisfied).

Let $\varepsilon > 0$. We denote by Λ_ε the open subset of $\mathbb{R} \times \Omega_\varepsilon \times \mathbb{R}^n$ defined by $\Lambda_\varepsilon = \{(t, x, p) \in \mathbb{R} \times \Omega \times \mathbb{R}^n \mid \forall s \in [0, t]\ \psi_1(s, x, p) \in \Omega_\varepsilon\}$, and we denote by $\mathcal{V}_{E,\varepsilon}$ the following smooth $2n-1$-dimensional submanifold of $\mathbb{R}^{2n}$ $\mathcal{V}_{E,\varepsilon} = \{(x, p) \in \Omega_\varepsilon \times \mathbb{R}^n \mid |p| = r_{\tilde V,E}(x)\}$. From (6.28) and continuity of $\partial_x^\alpha \tilde V$ and $\partial_x^{\alpha'} \tilde B$ for $\alpha, \alpha' \in (\mathbb{N} \cup \{0\})^n$, $|\alpha| \le 2$, $|\alpha'| \le 1$, and from (6.8)-(6.15), it follows that

$$\begin{aligned}
&C_i(\tilde V, \tilde B, \Omega_\varepsilon) \to C_i(\tilde V, \tilde B, D), \text{ as } \varepsilon \to 0^+, \text{ for } i = 1, 2,\\
&\sup_{x\in\Omega_\varepsilon} C_3(E, x, \tilde V, \tilde B, \Omega_\varepsilon) \to \sup_{x\in D} C_3(E, x, \tilde V, \tilde B, D), \text{ as } \varepsilon \to 0^+,\\
&C_i(E, \tilde V, \tilde B, \Omega_\varepsilon) \to C_i(E, \tilde V, \tilde B, D), \text{ as } \varepsilon \to 0^+, \text{ for } i = 6, 7, 8.
\end{aligned}$$

Taking also account of (6.29) and Propositions 6.2, 6.3, 6.4, we obtain that there exists $\varepsilon_0 > 0$ such that

$$\begin{aligned}
&\varphi_E^\varepsilon : \Lambda_\varepsilon \cap (]0, +\infty[\times \mathcal{V}_{E,\varepsilon}) \to \Omega_\varepsilon \times \Omega_\varepsilon, (t, x, p) \mapsto (x, \psi_1(t, x, p)), \text{ is a}\\
&C^1 \text{ diffeomorphism from } \Lambda_\varepsilon \cap (]0, +\infty[\times \mathcal{V}_{E,\varepsilon}) \text{ onto } (\Omega_\varepsilon \times \Omega_\varepsilon)\backslash\{(x, x) \mid\\
&x \in \Omega_\varepsilon\} \text{ for any } \varepsilon \in]0, \varepsilon_0[.
\end{aligned} \tag{6.30}$$

Let $q_0, q \in \bar D$, $q_0 \neq q$. Let $\varepsilon_1 \in]0, \varepsilon_0[$. From (6.30), it follows that there exists an unique $p_{\varepsilon_1} \in \mathbb{S}^{n-1}_{q_0,E}$ and an unique positive real number t_{ε_1} such that $q = \psi_1(t_{\varepsilon_1}, q_0, p_{\varepsilon_1})$ and $(t_{\varepsilon_1}, q_0, p_{\varepsilon_1}) \in \Lambda_{\varepsilon_1}$. Consider the function $m \in C^2(\mathbb{R}, \mathbb{R})$, defined by $m(t) = \chi_D(\psi_1(t, q_0, p_{\varepsilon_1}))$, $t \in \mathbb{R}$. Derivating twice m, we obtain

$$\begin{aligned}
\ddot m(t) &= Hess\chi_D(\psi_1(t, q_0, p_{\varepsilon_1}))(g(\psi_2(t, q_0, p_{\varepsilon_1})), g(\psi_2(t, q_0, p_{\varepsilon_1}))) \\
&+ \left(1 + \frac{|\psi_2(t, q_0, p_{\varepsilon_1})|^2}{c^2}\right)^{-1/2} \nabla\chi_D(\psi_1(t, q_0, p_{\varepsilon_1})) \circ F\,(\psi_1(t, q_0, p_{\varepsilon_1}),\\
&g(\psi_2(t, q_0, p_{\varepsilon_1}))) - \frac{\psi_2(t, q_0, p_{\varepsilon_1}) \circ F\,(\psi_1(t, q_0, p_{\varepsilon_1}), g(\psi_2(t, q_0, p_{\varepsilon_1})))}{c^2\left(1 + \frac{|\psi_2(t, q_0, p_{\varepsilon_1})|^2}{c^2}\right)^{3/2}}\\
&\times\nabla\chi_D(\psi_1(t, q_0, p_{\varepsilon_1})) \circ \psi_2(t, q_0, p_{\varepsilon_1}),
\end{aligned} \tag{6.31}$$

for $t \in \mathbb{R}$, where g is the function defined by (5.6) and $\circ$ denotes the usual scalar product on $\mathbb{R}^n$ (we used (5.1)). In addition, note that using the fact that χ_D is a C^2 defining function of D, we obtain that for $t \in \mathbb{R}$

$$\psi_1(t, q_0, p_{\varepsilon_1}) \in D \Leftrightarrow m(t) < 0,$$
$$\psi_1(t, q_0, p_{\varepsilon_1}) \in \partial D \Leftrightarrow m(t) = 0.$$

Assume that there exists some $s \in]0, t_{\varepsilon_1}[$ such that $\psi_1(s, q_0, p_{\varepsilon_1}) \notin D$ (i.e. $m(s) \geq 0$). Let $s_0 = \sup\{s' \in [0, s] \mid \psi_1(s', x, p_{\varepsilon_1}) \in \bar{D}\}$. Hence

$$\psi_1(s_0, q_0, p_{\varepsilon_1}) \in \partial D, \text{ (i.e. } m(s_0) = 0), \tag{6.32}$$
$$m(t) \leq 0, \text{ for } t \in [0, s_0]. \tag{6.33}$$

From (6.32), (6.31), (5.2), and the estimates (6.6), $|g(\psi_2(t, q_0, p_{\varepsilon_1}))| < c$, and definition (6.15), it follows that $\ddot{m}(s_0) \geq c^2 C_8(E, \tilde{V}, \tilde{B}, D) > 0$ (we used (6.29)). From (6.33) and from the estimate $\ddot{m}(s_0) > 0$ and Taylor expansion of m at s_0 ($m(t) = \dot{m}(s_0)(t - s_0) + \frac{1}{2}\ddot{m}(s_0)(t - s_0)^2 + o((t - s_0)^2),\ t \in \mathbb{R}$) it follows that $\dot{m}(s_0) > 0$. Using also the equality $m(s_0) = 0$, we obtain that there exists $\varepsilon' > 0$ such that $s_0 + \varepsilon' < t_{\varepsilon_1}$ and $m(s_0 + \varepsilon') > 0$ which implies that $\psi_1(s_0 + \varepsilon', q_0, p_{\varepsilon_1}) \notin \bar{D}$. Then, due to $\sup_{z \in \Omega_\varepsilon} \inf\{|z - z'| \mid z' \in \bar{D}\} \to 0$ as $\varepsilon \to 0$, there exists $\varepsilon_2 \in]0, \varepsilon_1[$ such that $\psi_1(s_0 + \varepsilon', q_0, p_{\varepsilon_1}) \notin \Omega_{\varepsilon_2}$ and using also (6.30), we obtain that there exists $(p_{\varepsilon_2}, t_{\varepsilon_2}) \in \mathbb{S}^{n-1}_{q_0,E} \times]0, +\infty[$ such that $(p_{\varepsilon_2}, t_{\varepsilon_2}) \neq (p_{\varepsilon_1}, t_{\varepsilon_1})$ and $(t_{\varepsilon_2}, q_0, p_{\varepsilon_2}) \in \Lambda_{\varepsilon_2}$ and $q = \psi_1(t_{\varepsilon_2}, q_0, p_{\varepsilon_2})$, which contradicts unicity of $(p_{\varepsilon_1}, t_{\varepsilon_1})$.

We finally proved that

$$\psi_1(s, q_0, p_{\varepsilon_1}) \in D \text{ for all } s \in]0, t_{\varepsilon_1}[. \tag{6.34}$$

Now consider

$$t_2 = \sup\{t \in]0, +\infty[\mid \psi_1(s, q_0, p_{\varepsilon_1}) \in D \text{ for all } s \in]0, t]\}, \tag{6.35}$$
$$t_1 = \inf\{t \in \mathbb{R} \mid \psi_1(s, q_0, p_{\varepsilon_1}) \in D \text{ for all } s \in [t, t_{\varepsilon_1}[\} \tag{6.36}$$

(using Proposition 6.1 and (6.29), we obtain that t_2 and t_1 are real numbers that satisfy $t_2 - t_1 \leq \frac{5\delta(D)}{c}$). Then for $i = 1, 2$, from (6.31), (5.2), (6.15) and (6.29), it follows that $\ddot{m}(t_i) \geq c^2 C_8(E, \tilde{V}, \tilde{B}, D) > 0$. For all $t \in]t_1, t_2[$, $\psi_1(t, q_0, p) \in D$. Hence $m(t) < 0,\ t \in]t_1, t_2[$. This latter estimate and the estimate $\ddot{m}(t_i) > 0$ and Taylor expansion of m at t_i ($m(t) = \dot{m}(t_i)(t - t_i) + \frac{1}{2}\ddot{m}(t_i)(t - t_i)^2 + o((t - t_i)^2),\ t \in \mathbb{R}$) for $i = 1, 2$, imply that $\dot{m}(t_2) > 0$ and $\dot{m}(t_1) < 0$, i.e.

$$\begin{aligned} &\tfrac{\partial \psi_1}{\partial t}(t, q_0, p_{\varepsilon_1})_{|t=t_1} \circ N(t_1) < 0, \\ &\tfrac{\partial \psi_1}{\partial t}(t, q_0, p_{\varepsilon_1})_{|t=t_2} \circ N(t_2) > 0, \end{aligned} \tag{6.37}$$

where $N(t_i) = \frac{\nabla\chi(\psi_1(t,q_0,p_{\varepsilon_1}))}{|\nabla\chi(\psi_1(t,q_0,p_{\varepsilon_1}))|}$, $i = 1, 2$.

Statement (6.34) with (6.37) and (6.30) (with "$\varepsilon = \varepsilon_1$") proves (2.1) and (2.2). $\square$

6.6 *Proof of Proposition 6.1.* We denote $\frac{\dot{x}(t)}{\sqrt{1-\frac{\dot{x}(t)^2}{c^2}}}$ by $p(t)$ for $t \in]t_-, t_+[$.

Let $I(t) = \frac{1}{2}|x(t)|^2$, for $t \in]t_-, t_+[$. Derivating twice I and using (6.2), we obtain

$$\begin{aligned} \ddot{I}(t) &= \frac{p(t)^2}{1+\frac{p(t)^2}{c^2}} + \frac{1}{\sqrt{1+\frac{p(t)^2}{c^2}}} F\left(x(t), \frac{p(t)}{\sqrt{1+\frac{p(t)^2}{c^2}}}\right) \circ x(t) \qquad (6.38) \\ &\quad - \frac{p(t)\circ x(t)}{c^2(1+\frac{p(t)^2}{c^2})^{3/2}} p(t) \circ F\left(x(t), \frac{p(t)}{\sqrt{1+\frac{p(t)^2}{c^2}}}\right) \end{aligned}$$

for $t \in]t_-, t_+[$, where $\circ$ denotes the usual scalar product in $\mathbb{R}^n$. From the estimate $\frac{|p(t)|}{\sqrt{1+\frac{p(t)^2}{c^2}}} < c$, $t \in]t_-, t_+[$, and from (6.38) and (5.2), it follows that $\ddot{I}(t) \geq c^2\left(1 - \frac{1}{(\frac{E-\tilde{V}(x(t))}{c^2})^2}\right) - 2\frac{|x(t)|\left(|\nabla\tilde{V}(x(t))| + \sum_{i,j=1\ldots n}|\tilde{B}_{i,j}(x(t))|\right)}{\frac{E-\tilde{V}(x(t))}{c^2}}$, for $t \in]t_-, t_+[$, which with (6.19) implies

$$\ddot{I}(t) \geq \frac{c^2}{2}, \qquad (6.39)$$

for $t \in]t_-, t_+[$.

Let $t, s \in]t_-, t_+[$, $s \leq t$. From (6.39) and the equality $I(t) = I(s) + \dot{I}(s)(t-s) + \int_s^t\int_s^\tau \ddot{I}(\sigma)d\sigma d\tau$, it follows that

$$I(t) - I(s) \geq \dot{I}(s)(t-s) + \frac{c^2}{4}(t-s)^2. \qquad (6.40)$$

Using (6.1) and the estimate $|\dot{x}(s)| < c$, we obtain $\dot{I}(s) = x(s) \circ \dot{x}(s) \geq -|x(s)||\dot{x}(s)| \geq -c\delta(\Omega)$. Using (6.1), we obtain $I(t) - I(s) = \frac{1}{2}(|x(t)|^2 - |x(s)|^2) \leq \delta(\Omega)^2$. From (6.40) and the two latter inequalities, it follows that $0 \geq -\delta(\Omega)^2 - c\delta(\Omega)(t-s) + \frac{c^2}{4}(t-s)^2$, which implies that $t - s \leq \frac{\delta(\Omega)}{c}(2\sqrt{2}+2) < \frac{5\delta(\Omega)}{c}$ (the roots of $-\delta(\Omega)^2 - c\delta(\Omega)X + \frac{c^2}{4}X^2$ are $(\delta(\Omega)/c)(2 \pm 2\sqrt{2})$). As $t \to t_+$ and $s \to t_-$, the latter inequality proves (6.20). Proposition 6.1 is proved. $\square$

6.7 *Proof of Proposition 6.2.* Throughout this Subsection, we denote by $\gamma_{x,p_i}(t)$ the point of $\mathbb{R}^n$ defined by

$$\gamma_{x,p_i}(t) = \psi_1(t, x, p_i),$$

for any $t \in \mathbb{R}$ and $i = 1, 2$, where $\psi = (\psi_1, \psi_2)$ is the flow of the differential system (5.1).

From (5.1), it follows that

$$\begin{aligned}\gamma_{x,p_i}(t) &= x + tv_i + \int_0^{t_i} \Big(g(p_i + \int_0^\sigma F(\gamma_{x,p_i}(\tau), \dot\gamma_{x,p_i}(\tau))d\tau) \\ &\quad -g(p_i)\big)\, d\sigma \qquad (6.41)\end{aligned}$$

for $t \in [0, t_{+,x,p_i}[$ and $i = 1, 2$, where t_{+,x,p_i} is defined by (6.21) for $i = 1, 2$.

From (6.41), it follows that

$$|t_1v_1 - t_2v_2| - \Delta(t_1, t_2) \le |\gamma_{x,p_1}(t_1) - \gamma_{x,p_2}(t_2)| \le |t_1v_1 - t_2v_2| + \Delta(t_1, t_2) \quad (6.42)$$

where

$$\begin{aligned}\Delta(t_1, t_2) = \Bigg|&\int_0^{t_1} \Big(g(p_1 + \int_0^\sigma F(\gamma_{x,p_1}(\tau), \dot\gamma_{x,p_1}(\tau))d\tau) - g(p_1)\Big)\, d\sigma \qquad (6.43)\\ &- \int_0^{t_2} \Big(g(p_2 + \int_0^\sigma F(\gamma_{x,p_2}(\tau), \dot\gamma_{x,p_2}(\tau))d\tau) - g(p_2)\Big)\, d\sigma\Bigg|,\end{aligned}$$

for $t_1 \in [0, t_{+,x,p_1}[$ and $t_2 \in [0, t_{+,x,p_2}[$. We shall look for an upper bound of $\Delta(t_1, t_2)$, $t_1 \in [0, t_{+,x,p_1}[$ and $t_2 \in [0, t_{+,x,p_2}[$, $t_2 \le t_1$.

First case: $v_1 \circ v_2 \le 0$. Using (5.8), we obtain that

$$\begin{aligned}&\left|\int_0^{t_i} \Big(g(p_i + \int_0^\sigma F(\gamma_{x,p_i}(\tau), \dot\gamma_{x,p_i}(\tau))d\tau) - g(p_i)\Big)\, d\sigma\right| \qquad (6.44)\\ &\le \sqrt{n} \int_0^{t_i} \sup_{\varepsilon \in [0,1]} \left(1 + c^{-2}\left|p_i + \varepsilon \int_0^\sigma F(\gamma_{x,p_i}(s), \dot\gamma_{x,p_i}(s))ds\right|^2\right)^{-1/2} \\ &\qquad\qquad \times \int_0^\sigma |F(\gamma_{x,p_i}(s), \dot\gamma_{x,p_i}(s))| ds d\sigma,\end{aligned}$$

for $i = 1, 2$. From (6.6) and (6.20) and from the estimate $|g(\psi_2(s, x, p_i))| \le c$, it follows that

$$\int_0^\sigma |F(\gamma_{x,p_i}(s), \dot\gamma_{x,p_i}(s))| ds \le \frac{10n\delta(\Omega)\beta(\tilde V, \tilde B, \Omega)}{c}. \qquad (6.45)$$

for $\sigma \in [0, t_i]$, $i = 1, 2$. Using $p_i \in \mathbb{S}^{n-1}_{x,E}$ and (6.45) and (6.23), we obtain

$$\left|p_i + \varepsilon \int_0^\sigma F(\gamma_{x,p_i}(s), \dot\gamma_{x,p_i}(s))ds\right| \ge \frac{1}{2} r_{\tilde V, E}(x), \qquad (6.46)$$

for $\varepsilon \in [0,1]$ and $\sigma \in [0,t_i]$. From (6.45), (6.46) and (6.44), it follows that

$$\begin{aligned} &\left|\int_0^{t_i} \left(g(p_i + \int_0^{\sigma} F(\gamma_{x,p_i}(s), \dot{\gamma}_{x,p_i}(s))ds) - g(p_i)\right) d\sigma\right| \\ \leq\ & t_i \frac{20n^{3/2} c\delta(\Omega)\beta(\tilde{V}, \tilde{B}, \Omega)}{E - \tilde{V}(x)}, \end{aligned} \tag{6.47}$$

for $i = 1, 2$. Using $v_1 \circ v_2 \leq 0$, we obtain that $|v_i| s_i \leq |s_1 v_1 - s_2 v_2|$ for $i = 1, 2$ and for $s_1 \geq 0$, $s_2 \geq 0$. Using this latter inequality and (6.43) and (6.47) and equality $|v_i| = c\sqrt{1 - \left(\frac{E - \tilde{V}(x)}{c^2}\right)^{-2}}$, we obtain

$$\Delta(t_1, t_2) \leq 40 c^{-1} n^{3/2} \delta(\Omega)\beta(\tilde{V}, \tilde{B}, \Omega) r_{\tilde{V},E}(x)^{-1} |t_1 v_1 - t_2 v_2|.$$

Second case: $v_1 \circ v_2 \geq 0$.
From (6.43), it follows that

$$\Delta(t_1, t_2) \leq \Delta_1(t_1, t_2) + \Delta_2(t_1, t_2), \tag{6.48}$$

where

$$\Delta_1(t_1, t_2) = \left|\int_{t_2}^{t_1} \left(g(p_1 + \int_0^{\sigma} F(\gamma_{x,p_1}(\tau), \dot{\gamma}_{x,p_1}(\tau))d\tau) - g(p_1)\right) d\sigma\right| \tag{6.49}$$

$$\begin{aligned} \Delta_2(t_1, t_2) = & \left|\int_0^{t_2} \left[g(p_1 + \int_0^{\sigma} F(\gamma_{x,p_1}(\tau), \dot{\gamma}_{x,p_1}(\tau))d\tau) - g(p_1) d\sigma\right.\right. \\ & \left.\left. - \left(g(p_2 + \int_0^{\sigma} F(\gamma_{x,p_2}(\tau), \dot{\gamma}_{x,p_2}(\tau))d\tau) - g(p_2)\right)\right] d\sigma\right|. \end{aligned} \tag{6.50}$$

An upper bound for $\Delta_1(t_1, t_2)$. Using (6.49) and (5.8), we obtain that

$$\begin{aligned} \Delta_1(t_1, t_2) \leq & \sqrt{n}(t_1 - t_2) \int_{t_2}^{t_1} |F(\gamma_{x,p_1}(s), \dot{\gamma}_{x,p_1}(s))| ds \\ \times & \sup_{\substack{\varepsilon \in [0,1] \\ \sigma \in [0,t_1]}} \left(1 + c^{-2}\left|p_1 + \varepsilon \int_0^{\sigma} F(\gamma_{x,p_1}(s), \dot{\gamma}_{x,p_1}(s))ds\right|^2\right)^{-1/2}. \end{aligned} \tag{6.51}$$

In the same manner than in the first case ($v_1 \circ v_2 \leq 0$), we obtain

$$\Delta_1(t_1, t_2) \leq (t_1 - t_2)\frac{20n^{3/2} c\delta(\Omega)\beta(\tilde{V}, \tilde{B}, \Omega)}{E - \tilde{V}(x)}. \tag{6.52}$$

Note that from $|v_1| = |v_2|$ and $t_i \geq 0, i = 1, 2$, it follows that $|v_1|(t_1 - t_2) \leq |t_1 v_1 - t_2 v_2|$. Using this latter inequality with (6.52), we obtain

$$\Delta_1(t_1, t_2) \leq \frac{20 n^{3/2} \delta(\Omega) \beta(\tilde{V}, \tilde{B}, \Omega)}{c r_{\tilde{V},E}(x)} |t_1 v_1 - t_2 v_2| \tag{6.53}$$

(we use the equality $|v_1| = c\sqrt{1 - \left(\frac{E - \tilde{V}(x)}{c^2}\right)^{-2}}$).

An upper bound for $\Delta_2(t_1, t_2)$. Note that $g_j(p_i + \int_0^\sigma F(\gamma_{x,p_i}(\tau), \dot{\gamma}_{x,p_i}(\tau)) d\tau) - g_j(p_i) = \int_0^1 \nabla g_j(p_i + \varepsilon \int_0^\sigma F(\gamma_{x,p_i}(\tau), \dot{\gamma}_{x,p_i}(\tau)) d\tau) \circ \int_0^\sigma F(\gamma_{x,p_i}(s), \dot{\gamma}_{x,p_i}(s)) ds\, d\varepsilon$ for $i = 1, 2$ and $j = 1..n$, where $g = (g_1, \ldots, g_n)$. Hence

$$\Delta_2^j(t_1, t_2) \leq \int_0^{t_2} \Delta_{2,1,j}(\sigma) d\sigma + \int_0^{t_2} \Delta_{2,2,j}(\sigma) d\sigma, \tag{6.54}$$

where $\Delta_2(t_1, t_2) = (\Delta_2^1(t_1, t_2), \ldots, \Delta_2^n(t_1, t_2))$ and

$$\Delta_{2,1,j}(\sigma) = \left| \int_0^1 \left[\nabla g_j(p_1 + \varepsilon \int_0^\sigma F(\gamma_{x,p_1}(\tau), \dot{\gamma}_{x,p_1}(\tau)) d\tau) - \nabla g_j \right.\right. \tag{6.55}$$

$$\left.\left. (p_2 + \varepsilon \int_0^\sigma F(\gamma_{x,p_2}(\tau), \dot{\gamma}_{x,p_2}(\tau)) d\tau) \right] \circ \int_0^\sigma F(\gamma_{x,p_1}(s), \dot{\gamma}_{x,p_1}(s)) ds d\varepsilon \right|,$$

$$\Delta_{2,2,j}(\sigma) = \left| \int_0^1 \nabla g_j(p_2 + \varepsilon \int_0^\sigma F(\gamma_{x,p_2}(\tau), \dot{\gamma}_{x,p_2}(\tau)) d\tau) \circ \right. \tag{6.56}$$

$$\left. \int_0^\sigma \left[F(\gamma_{x,p_1}(s), \dot{\gamma}_{x,p_1}(s)) - F(\gamma_{x,p_2}(s), \dot{\gamma}_{x,p_2}(s)) \right] ds d\varepsilon \right|$$

for $\sigma \in [0, t_2]$ and $j = 1 \ldots n$.

We first look for an upper bound for $\Delta_{2,1,j}(\sigma)$. Since $v_1 \circ v_2 \geq 0$, we obtain $p_1 \circ p_2 \geq 0$. Using this latter inequality and the equality $p_1^2 = p_2^2$, we obtain that

$$\begin{aligned} |\mu p_1 + (1 - \mu) p_2| &= \sqrt{\mu^2 p_1^2 + (1 - \mu)^2 p_2^2 + 2\mu(1 - \mu) p_1 \circ p_2} \\ &\geq \sqrt{\mu^2 p_1^2 + (1 - \mu)^2 p_2^2} \geq \frac{1}{\sqrt{2}} |p_1| = \frac{r_{\tilde{V},E}(x)}{\sqrt{2}}, \end{aligned} \tag{6.57}$$

for any $\mu \in [0, 1]$.

From (6.45) and (6.57) and (6.23), it follows that

$$\begin{aligned} &\left| \mu p_1 + (1 - \mu) p_2 + \mu \epsilon \int_0^\sigma F(\gamma_{x,p_1}(\tau), \dot{\gamma}_{x,p_1}(\tau)) d\tau + (1 - \mu) \epsilon \right. \\ &\left. \times \int_0^\sigma F(\gamma_{x,p_2}(\tau), \dot{\gamma}_{x,p_2}(\tau)) d\tau \right| \geq |\mu p_1 + (1 - \mu) p_2| - \frac{10 n \delta(\Omega) \beta(\tilde{V}, \tilde{B}, \Omega)}{c} \\ &\geq \frac{1}{2\sqrt{2}} r_{\tilde{V},E}(x), \end{aligned} \tag{6.58}$$

for μ, $\epsilon \in [0,1]$. From (5.1), it follows that

$$\dot\gamma_{x,p_i}(\sigma) = g(p_i + \int_0^\sigma F(\gamma_{x,p_i}(\tau), \dot\gamma_{x,p_i}(\tau))d\tau)$$

for $\sigma \in [0, t_i]$, $i = 1, 2$. Using this latter equality and (5.8), we obtain

$$\begin{aligned} &|\dot\gamma_{x,p_1}(\sigma) - \dot\gamma_{x,p_2}(\sigma)| \leq \qquad (6.59)\\ &\sqrt{n} \sup_{\mu\in[0,1]} \Big(1 + c^{-2}|\mu p_1 + (1-\mu)p_2 + \mu \int_0^\sigma F(\gamma_{x,p_1}(\tau), \dot\gamma_{x,p_1}(\tau))d\tau \\ &+ (1-\mu) \int_0^\sigma F(\gamma_{x,p_2}(\tau), \dot\gamma_{x,p_2}(\tau))d\tau|^2\Big)^{-1/2} \\ &\times |p_1 - p_2 + \int_0^\sigma \left(F(\gamma_{x,p_1}(\tau), \dot\gamma_{x,p_1}(\tau)) - F(\gamma_{x,p_2}(\tau), \dot\gamma_{x,p_2}(\tau))\right) d\tau| \end{aligned}$$

for $\sigma \in [0, t_2]$.

Note that from (6.7) and the inequality $|\dot\gamma_{x,p_2}(\tau)| \leq c$, it follows that

$$\begin{aligned} &|F(\gamma_{x,p_1}(\tau), \dot\gamma_{x,p_1}(\tau)) - F(\gamma_{x,p_2}(\tau), \dot\gamma_{x,p_2}(\tau))| \leq \qquad (6.60)\\ &n\beta(\tilde V, \tilde B, \Omega)(2|\gamma_{x,p_1}(\tau) - \gamma_{x,p_2}(\tau)| + \frac{1}{c}|\dot\gamma_{x,p_1}(\tau) - \dot\gamma_{x,p_2}(\tau)|), \end{aligned}$$

for $\tau \in [0, t_2]$.

Using (6.58)-(6.60), we obtain

$$\begin{aligned} |\dot\gamma_{x,p_1}(\sigma) - \dot\gamma_{x,p_2}(\sigma)| \leq \frac{2^{3/2}\sqrt{n}c^2}{E - \tilde V(x)} \Big[|p_1 - p_2| + n\beta(\tilde V, \tilde B, \Omega) \qquad (6.61)\\ \left(2\int_0^\sigma |\gamma_{x,p_1}(\tau) - \gamma_{x,p_2}(\tau)|d\tau + \frac{1}{c}\int_0^\sigma |\dot\gamma_{x,p_1}(\tau) - \dot\gamma_{x,p_2}(\tau)|d\tau\right)\Big], \end{aligned}$$

for $\sigma \in [0, t_2]$.

We shall use the following Gronwall's lemma.

Gronwall's lemma. *Let $a > 0$ and let $\phi \in C([0,a], [0,+\infty[)$ be a continuous map and let $A, B \in [0,+\infty[$ be such that $\phi(t) \leq A + B\int_0^t \phi(s)ds$ for all $t \in [0,a]$. Then $\phi(t) \leq Ae^{Bt}$ for all $t \in [0,a]$.*

Taking account of (6.61), Gronwall's lemma and (6.20), we obtain that

$$\begin{aligned} |\dot\gamma_{x,p_1}(\sigma) - \dot\gamma_{x,p_2}(\sigma)| &\leq \frac{2^{3/2}\sqrt{n}c^2}{E - \tilde V(x)} \Big[|p_1 - p_2| + 2n\beta(\tilde V, \tilde B, \Omega) \qquad (6.62)\\ &\int_0^\sigma |\gamma_{x,p_1}(\tau) - \gamma_{x,p_2}(\tau)|d\tau\Big] e^{\frac{10\sqrt{2}\delta(\Omega)n^{3/2}\beta(\tilde V, \tilde B, \Omega)}{E - \tilde V(x)}}, \end{aligned}$$

for $\sigma \in [0, t_2]$.

From (5.9) and (6.55), it follows that

$$\Delta_{2,1,j}(\sigma) \leq \tag{6.63}$$
$$\frac{3\sqrt{n}}{c}\int_0^1 \sup_{\mu\in[0,1]}\left(1+\frac{1}{c^2}\left|\mu p_1+(1-\mu)p_2+\mu\epsilon\int_0^\sigma F(\gamma_{x,p_1}(\tau),\dot\gamma_{x,p_1}(\tau))d\tau\right.\right.$$
$$\left.\left.+\ (1-\mu)\epsilon\int_0^\sigma F(\gamma_{x,p_2}(\tau),\dot\gamma_{x,p_2}(\tau))d\tau\right|^2\right)^{-1}$$
$$\times|p_1-p_2+\varepsilon\int_0^\sigma\left(F(\gamma_{x,p_1}(\tau),\dot\gamma_{x,p_1}(\tau))-F(\gamma_{x,p_2}(\tau),\dot\gamma_{x,p_2}(\tau))\right)d\tau|$$
$$\times\int_0^\sigma|F(\gamma_{x,p_1}(s),\dot\gamma_{x,p_1}(s))|dsd\varepsilon,$$

for $\sigma \in [0, t_2]$.

From (6.6) and the estimate $|\dot\gamma_{x,p_1}(s)| \leq c$, it follows that

$$\int_0^\sigma|F(\gamma_{x,p_1}(s),\dot\gamma_{x,p_1}(s))|ds \leq 2n\beta(\tilde V,\tilde B,\Omega)\sigma. \tag{6.64}$$

for $\sigma \in [0, t_1]$.

From (6.63), (6.58), (6.64) and (6.60) and (6.62), it follows that

$$\Delta_{2,1,j}(\sigma) \leq \frac{48n^{3/2}c^3\beta(\tilde V,\tilde B,\Omega)}{(E-\tilde V(x))^2}\Big(|p_1-p_2|\sigma+n\beta(\tilde V,\tilde B,\Omega)\sigma \tag{6.65}$$
$$\Big(2\int_0^\sigma|\gamma_{x,p_1}(\tau)-\gamma_{x,p_2}(\tau)|d\tau+\frac{2^{3/2}\sqrt{n}c}{E-\tilde V(x)}\Big(|p_1-p_2|\sigma+2n\beta(\tilde V,\tilde B,\Omega)$$
$$\int_0^\sigma\int_0^s|\gamma_{x,p_1}(\tau)-\gamma_{x,p_2}(\tau)|d\tau ds\Big)\times e^{\frac{10\sqrt{2}\delta(\Omega)n^{3/2}\beta(\tilde V,\tilde B,\Omega)}{E-\tilde V(x)}}\Big)\Big)$$

for $\sigma \in [0, t_2]$.

From $|v_1| = |v_2|$ and $t_2 \leq t_1$, it follows that $|v_1 - v_2|\sigma \leq |v_1 - v_2|t_2 \leq |t_1v_1 - t_2v_2|$, for $\sigma \in [0, t_2]$. Note that using these latter estimates and $p_i \in \mathbb{S}^{n-1}_{x,E}$, $i = 1, 2$, we obtain

$$|p_1-p_2|\sigma \leq \frac{E-\tilde V(x)}{c^2}|t_1v_1-t_2v_2|, \tag{6.66}$$

for $\sigma \in [0, t_2]$.

Using (6.66), the estimate $\int_0^\sigma\int_0^s|\gamma_{x,p_1}(\tau)-\gamma_{x,p_2}(\tau)|d\tau ds\le\sigma\int_0^\sigma|\gamma_{x,p_1}(\tau)-\gamma_{x,p_2}(\tau)|d\tau$ and $\sigma\le\frac{5\delta(\Omega)}{c}$ (due to (6.20)) and (6.65), we obtain

$$\begin{aligned}\Delta_{2,1,j}(\sigma)\le&\left(1+\frac{10\sqrt{2}n^{3/2}\delta(\Omega)\beta(\tilde V,\tilde B,\Omega)}{E-\tilde V(x)}e^{\frac{10\sqrt{2}\delta(\Omega)n^{3/2}\beta(\tilde V,\tilde B,\Omega)}{E-\tilde V(x)}}\right)\\ &\times\left(\frac{48n^{3/2}c\delta(\Omega)\beta(\tilde V,\tilde B,\Omega)}{E-\tilde V(x)}|t_1v_1-t_2v_2|\right.\\ &\left.+\frac{480n^{5/2}c^2\delta(\Omega)\beta(\tilde V,\tilde B,\Omega)^2}{(E-\tilde V(x))^2}\int_0^\sigma|\gamma_{x,p_1}(\tau)-\gamma_{x,p_2}(\tau)|d\tau\right)\end{aligned}\tag{6.67}$$

for $\sigma\in[0,t_2]$.

We look for an upper bound for $\Delta_{2,2,j}(\sigma)$, $\sigma\in[0,t_2]$, defined by (6.56). Using (6.56), (5.7), and (6.46), and using (6.60), (6.62) and (6.66), we obtain

$$\begin{aligned}\Delta_{2,2,j}(\sigma)\ \le\ &\left(1+\frac{10\sqrt{2}n^{3/2}\delta(\Omega)\beta(\tilde V,\tilde B,\Omega)}{E-\tilde V(x)}e^{\frac{10\sqrt{2}\delta(\Omega)n^{3/2}\beta(\tilde V,\tilde B,\Omega)}{E-\tilde V(x)}}\right)\\ &\times\frac{4nc^2\beta(\tilde V,\tilde B,\Omega)}{E-\tilde V(x)}\int_0^\sigma|\gamma_{x,p_1}(\tau)-\gamma_{x,p_2}(\tau)|d\tau\\ &+\frac{2^{5/2}n^{3/2}c\beta(\tilde V,\tilde B,\Omega)}{E-\tilde V(x)}e^{\frac{10\sqrt{2}\delta(\Omega)n^{3/2}\beta(\tilde V,\tilde B,\Omega)}{E-\tilde V(x)}}|t_1v_1-t_2v_2|,\end{aligned}\tag{6.68}$$

for $\sigma\in[0,t_2]$.

Note also that from (6.42), it follows that $\int_0^\sigma|\gamma_{x,p_1}(\tau)-\gamma_{x,p_2}(\tau)|d\tau\le\int_0^\sigma\Delta(\tau,\tau)d\tau+\frac{\sigma^2}{2}|v_1-v_2|$, for $\sigma\in[0,t_2]$. Hence using also $\sigma\le\frac{5\delta(\Omega)}{c}$ (due to (6.20)) and $\sigma|v_1-v_2|\le|t_1v_1-t_2v_2|$, we obtain

$$\int_0^\sigma|\gamma_{x,p_1}(\tau)-\gamma_{x,p_2}(\tau)|d\tau\le\int_0^\sigma\Delta(\tau,\tau)d\tau+\frac{5\delta(\Omega)}{2c}|t_1v_1-t_2v_2|,\tag{6.69}$$

for $\sigma\in[0,t_2]$. Note that $t_2\le\frac{5\delta(\Omega)}{c}$ and note that from positiveness of Δ it follows that $\int_0^{t_2}\int_0^\sigma\Delta(\tau,\tau)d\tau d\sigma\le t_2\int_0^{t_2}\Delta(\tau,\tau)d\tau$. Hence using (6.69), we obtain

$$\int_0^{t_2}\int_0^\sigma|\gamma_{x,p_1}(\tau)-\gamma_{x,p_2}(\tau)|d\tau d\sigma\le\frac{5\delta(\Omega)}{c}\int_0^{t_2}\Delta(\tau,\tau)d\tau+\frac{25\delta(\Omega)^2}{2c^2}|t_1v_1-t_2v_2|,\tag{6.70}$$

for $\sigma\in[0,t_2]$.

Combining (6.48), (6.53), (6.67), (6.68) and (6.70), we obtain

$$\Delta(t_1,t_2)\le C_4(E,x,\tilde V,\tilde B,\Omega)|t_1v_1-t_2v_2|+cC_5(E,x,\tilde V,\tilde B,\Omega)\int_0^{t_2}\Delta(\tau,\tau)d\tau,\tag{6.71}$$

for $t_1 \in [0, t_{+,x,p_1}[$ and $t_2 \in [0, t_{+,x,p_2}[$, $t_1 \geq t_2$ and where C_4 and C_5 are defined by (6.11) and (6.12).

Let $t_1 \in [0, t_{+,x,p_1}[$ and $t_2 \in [0, t_{+,x,p_2}[$, $t_1 \geq t_2$. Estimates (6.71) and $|v_1 - v_2|\sigma \leq |t_1 v_1 - t_2 v_2|$, $\sigma \leq t_2$, give in particular

$$\Delta(\sigma,\sigma) \;\leq\; C_4|t_1v_1 - t_2v_2| + cC_5\int_0^\sigma \Delta(\tau,\tau)d\tau \tag{6.72}$$

for $\sigma \in [0, t_2]$. Using (6.72) and using Gronwall's lemma (formulated above) and $\sigma \leq \frac{5\delta(\Omega)}{c}$, we obtain

$$\Delta(\sigma,\sigma) \leq C_4 e^{5\delta(\Omega)C_5}|t_1v_1 - t_2v_2|, \tag{6.73}$$

for $\sigma \in [0, t_2]$.

Using (6.73) and (6.71) and $t_2 \leq \frac{5\delta(\Omega)}{c}$, we obtain $\Delta(t_1, t_2) \leq C_3(E, x, \tilde{V}, \tilde{B}, \Omega)|t_1v_1 - t_2v_2|$, for $t_1 \in [0, t_{+,x,p_1}[$ and $t_2 \in [0, t_{+,x,p_2}[$, $t_1 \geq t_2$.

Proposition 6.2 is proved. □

6.8 *Proof of Proposition 6.3.* We shall work in coordinates. We consider the following infinitely smooth parametrizations of $\mathbb{S}^{n-1}$, $\phi_{i,\pm} : B_{n-1}(0,1) \to \mathbb{S}^{n-1}$, $i = 1 \ldots n$, defined by

$$\phi_{i,\pm}(w) = \tag{6.74}$$
$$\begin{cases} \left(w^1, \ldots, w^{i-1}, \pm\sqrt{1 - \sum_{l=1}^{n-1} {w^l}^2}, w^i, \ldots, w^{n-1}\right), & \text{if } 1 \leq i \leq n-1, \\ \left(w^1, \ldots, w^{n-1}, \pm\sqrt{1 - \sum_{l=1}^{n-1} {w^l}^2}\right), & \text{if } i = n \end{cases}$$

for $w = (w^1, \ldots, w^{n-1}) \in B_{n-1}(0,1)$ and where $B_{n-1}(0,1)$ denotes the unit Euclidean open ball of $\mathbb{R}^{n-1}$ of center 0.

Let $(t_0, x_0, p_0) \in \Lambda \cap (]0, +\infty[\times \mathcal{V}_E)$, $p_0 = (p_0^1, \ldots, p_0^n)$. Then $(t, x_0, p_0) \in \Lambda$ for all $t \in [0, t_0]$. As Λ is an open subset of $\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^n$, there exists $\varepsilon > 0$ such that $\{(t, x, p) \in \mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^n \mid -\varepsilon < t < t_0+\varepsilon,\ \max(|x-x_0|, |p-p_0|) < \varepsilon\} \subseteq \Lambda$. We denote by $B(x_0, \varepsilon)$ the Euclidean open ball of $\mathbb{R}^n$ of center x_0 and radius ε. Let (U, ϕ) be an infinitely smooth parametrization of an open neighborhood of $\frac{p_0}{|p_0|}$ in $\mathbb{S}^{n-1}$, and $k = 1 \ldots n$ such that

$$U \text{ is an open subset of } B_{n-1}(0,1), \tag{6.75}$$
$$|p_0^k| \geq n^{-1/2}|p_0|, \tag{6.76}$$
$$\text{if } \pm p_0^k > 0 \text{ then } \phi(w) = \phi_{k,\pm}(w) \text{ for all } w \in U, \tag{6.77}$$

$$(t,x,r_{\tilde{V},E}(x)\phi(w))\in\Lambda,\ \text{ for } (w,t,x)\in U\times]-\varepsilon,t_0+\varepsilon[\times B(x_0,\varepsilon). \tag{6.78}$$

Consider $Q\in C^1(]-\varepsilon,t_0+\varepsilon[\times B(x_0,\varepsilon)\times U,\Omega)$ defined by

$$Q(t,x,w)=\psi_1(t,x,r_{\tilde{V},E}(x)\phi(w)),\ (t,x,w)\in]-\varepsilon,t_0+\varepsilon[\times B(x_0,\varepsilon)\times U, \tag{6.79}$$

where $\psi=(\psi_1,\psi_2)$ is the flow of the differential system (5.1). Let $w_0\in U$ be such that $\phi(w_0)=\frac{p_0}{|p_0|}$. We shall prove that $(\frac{\partial Q}{\partial t}(t_0,x_0,w_0),\frac{\partial Q}{\partial w^1}(t_0,x_0,w_0),\ldots,\frac{\partial Q}{\partial w^{n-1}}(t_0,x_0,w_0))$ is a basis of $\mathbb{R}^n$.

Note that from (6.2), it follows that

$$\begin{aligned}Q(t,x,w)&=x+tg(r_{\tilde{V},E}(x)\phi(w))+\int_0^t\Big[g\Big(r_{\tilde{V},E}(x)\phi(w)\\&+\int_0^\sigma F(Q(s,x,w),\frac{\partial Q}{\partial s}(s,x,w))ds\Big)-g(r_{\tilde{V},E}(x)\phi(w))\Big]d\sigma,\end{aligned} \tag{6.80}$$

for $w\in U$ and $(t,x)\in]-\varepsilon,t_0+\varepsilon[\times B(x_0,\varepsilon)$ (where g, $r_{\tilde{V},E}$ and F are defined by (5.6), (5.4) and (5.3)).

We shall prove (6.82).

Using (6.80) we obtain

$$\begin{aligned}\frac{\partial Q}{\partial t}(t,x,w)&=g(r_{\tilde{V},E}(x)\phi(w))+\Big[g\Big(r_{\tilde{V},E}(x)\phi(w)\\&\quad+\int_0^t F(Q(s,x,w),\frac{\partial Q}{\partial s}(s,x,w))ds\Big)-g(r_{\tilde{V},E}(x)\phi(w))\Big],\end{aligned} \tag{6.81}$$

for $w\in U$ and $(t,x)\in]-\varepsilon,t_0+\varepsilon[\times B(x_0,\varepsilon)$. Combining (6.81), (6.26), (5.8), (6.6) and estimates $|\frac{\partial Q}{\partial t}(t,x,w)|\le c$, $t\le\frac{5\delta(\Omega)}{c}$, it follows that

$$\left|\frac{\partial Q}{\partial t}(t,x,w)-g(r_{\tilde{V},E}(x)\phi(w))\right|\le\frac{4n^{3/2}c^2\beta(\tilde{V},\tilde{B},\Omega)}{E-\tilde{V}(x)}t, \tag{6.82}$$

for $w\in U$ and $(t,x)\in[0,t_0+\varepsilon[\times B(x_0,\varepsilon)$.

We shall prove (6.93). Let $i=1\ldots n-1$. Let $X_i\in C(]-\varepsilon,t_0+\varepsilon[\times B(x_0,\varepsilon)\times U,\mathbb{R}^n)$ be defined by

$$\begin{aligned}X_i^j(s,x,w)=\sum_{l=1}^n\Big(&\frac{\partial F_j}{\partial x'_l}(x',\frac{\partial Q}{\partial s}(s,x,w))_{|x'=Q(s,x,w)}\frac{\partial Q_l}{\partial w^i}(s,x,w)\\&+\frac{\partial F_j}{\partial y'_l}(Q(s,x,w),y')_{|y'=\frac{\partial Q}{\partial s}(s,x,w)}\frac{\partial\bar{Q}_l}{\partial w^i}(s,x,w)\Big),\end{aligned} \tag{6.83}$$

for $j = 1\dots n$, $(s,x,w) \in]-\varepsilon, t_0+\varepsilon[\times B(x_0,\varepsilon) \times U$, and where $X_i = (X_i^1,\dots,X_i^n)$ and $\bar{Q} \in C^1(]-\varepsilon,t_0+\varepsilon[\times B(x_0,\varepsilon)\times U,\mathbb{R})$ is defined by

$$\bar{Q}(s,x,w) = g(\psi_2(s,x,r_{\tilde{V},E}(x)\phi(w))) = \frac{\partial Q}{\partial s}(s,x,w), \tag{6.84}$$

for $(s,x,w) \in]-\varepsilon,t_0+\varepsilon[\times B(x_0,\varepsilon)\times U$.

From (6.5), and (5.3), it follows that

$$|X_i(\sigma,x,w)| \leq \beta(\tilde{V},\tilde{B},\Omega)n\left(2\sqrt{n}\left|\frac{\partial Q}{\partial w^i}(\sigma,x,w)\right| + c^{-1}\left|\frac{\partial \bar{Q}}{\partial w^i}(\sigma,x,w)\right|\right), \tag{6.85}$$

for $(\sigma,x,w) \in]-\varepsilon,t_0+\varepsilon[\times B(x_0,\varepsilon)\times U$.

We shall estimate $\bar{Q}$. Note that from (6.2), it follows that $\bar{Q}_l(s,x,w) = g_l(r_{\tilde{V},E}(x)\phi(w) + \int_0^s F(Q(\sigma,x,r_{\tilde{V},E}(x)\phi(w)),\bar{Q}(\sigma,x,r_{\tilde{V},E}(x)\phi(w)))d\sigma)$, for $(s, x,w) \in]-\varepsilon,t_0+\varepsilon[\times B(x_0,\varepsilon)\times U$ and $l = 1\dots n$. From this latter equality and (6.83), it follows that

$$\begin{aligned}\frac{\partial \bar{Q}_l}{\partial w^i}(s,x,w) = \nabla g_l \bigg(& r_{\tilde{V},E}(x)\phi(w) + \int_0^s F(Q(\sigma,x,r_{\tilde{V},E}(x)\phi(w)),\\ & \bar{Q}(\sigma,x,r_{\tilde{V},E}(x)\phi(w)))d\sigma\bigg) \circ \left(r_{\tilde{V},E}(x)\frac{\partial\phi}{\partial w^i}(w) + \int_0^s X_i(\sigma,x,w)d\sigma\right),\end{aligned}$$

for $(s,x,w) \in]-\varepsilon,t_0+\varepsilon[\times B(x_0,\varepsilon)\times U$. Hence

$$\begin{aligned}&\left|\frac{\partial \bar{Q}_j}{\partial w^i}(s,x,w) - r_{\tilde{V},E}(x)\nabla g_j(r_{\tilde{V},E}(x)\phi(w)) \circ \frac{\partial\phi}{\partial w^i}(w)\right| \leq \qquad (6.86)\\ &r_{\tilde{V},E}(x)\left|\left(\nabla g_j\left(r_{\tilde{V},E}(x)\phi(w) + \int_0^s F(Q(\sigma,x,r_{\tilde{V},E}(x)\phi(w)),\right.\right.\right.\\ &\left.\left.\bar{Q}(\sigma,x,r_{\tilde{V},E}(x)\phi(w)))d\sigma\right) - \nabla g_j\left(r_{\tilde{V},E}(x)\phi(w)\right)\right) \circ \frac{\partial\phi}{\partial w^i}(w)\Bigg|\\ &+\left|\nabla g_j\left(r_{\tilde{V},E}(x)\phi(w) + \int_0^s F(Q(\sigma,x,r_{\tilde{V},E}(x)\phi(w)),\bar{Q}(\sigma,x,r_{\tilde{V},E}(x)\right.\right.\\ &\left.\left.\phi(w)))d\sigma\right) \circ \int_0^s X_i(\sigma,x,w)d\sigma\right|,\end{aligned}$$

for $j = 1\dots n$ and $(s,x,w) \in]-\varepsilon,t_0+\varepsilon[\times B(x_0,\varepsilon)\times U$. We estimate the second term of the sum on the right-hand side of (6.86) by using (5.7), (6.6), (6.26) and $s \leq \frac{5\delta(\Omega)}{c}$, and (6.85). We estimate the first term of the sum on the right-hand side of (6.86) by using (5.9) and (6.6), (6.26) and $s \leq$

$\frac{5\delta(\Omega)}{c}$, and the estimate $\int_0^s F(Q(\sigma, x, r_{\tilde{V},E}(x)\phi(w)), \bar{Q}(\sigma, x, r_{\tilde{V},E}(x)\phi(w)))d\sigma \leq 2n\beta(\tilde{V}, \tilde{B}, \Omega)s$, for $(s, x, w) \in]-\varepsilon, t_0+\varepsilon[\times B(x_0, \varepsilon) \times U$. We obtain

$$\left|\frac{\partial \bar{Q}}{\partial w^i}(s, x, w) - \frac{r_{\tilde{V},E}(x)}{\left(\frac{E-\tilde{V}(x)}{c^2}\right)}\frac{\partial \phi}{\partial w^i}(w)\right| \leq 2n\sqrt{n}c^3\beta(\tilde{V}, \tilde{B}, \Omega)\left(12\sqrt{n}+1\right)$$

$$\times \frac{r_{\tilde{V},E}(x)}{\left(E-\tilde{V}(x)\right)^2}|\frac{\partial \phi}{\partial w^i}(w)|s + \frac{2c^2\beta(\tilde{V}, \tilde{B}, \Omega)n\sqrt{n}}{E-\tilde{V}(x)}\left(\int_0^s \left|\frac{\partial Q}{\partial w^i}(\sigma, x, w)\right| d\sigma\right.$$

$$\left.\times 2\sqrt{n} + c^{-1}\int_0^s \left|\frac{\partial \bar{Q}}{\partial w^i}(\sigma, x, w) - \frac{r_{\tilde{V},E}(x)}{\left(\frac{E-\tilde{V}(x)}{c^2}\right)}\frac{\partial \phi}{\partial w^i}(w)\right| d\sigma\right),$$

for $(s, x, w) \in [0, t_0+\varepsilon[\times B(x_0, \varepsilon) \times U$ (note that $\left(\frac{E-\tilde{V}(x)}{c^2}\right)^{-1}\frac{\partial \phi}{\partial w^i}(w) = \left(\nabla g_j(r_{\tilde{V},E}(x)\phi(w)) \circ \frac{\partial \phi}{\partial w^i}(w)\right)_{j=1\ldots n}$).

From Gronwall's lemma (formulated in Subsection 6.7) and $t_0+\varepsilon \leq \frac{5\delta(\Omega)}{c}$, it follows that

$$\left|\frac{\partial \bar{Q}}{\partial w^i}(s, x, w) - \frac{r_{\tilde{V},E}(x)}{\left(\frac{E-\tilde{V}(x)}{c^2}\right)}\frac{\partial \phi}{\partial w^i}(w)\right| \leq \frac{2c^2n^{3/2}\beta(\tilde{V}, \tilde{B}, \Omega)}{E-\tilde{V}(x)}e^{\frac{10\delta(\Omega)\beta(\tilde{V}, \tilde{B}, \Omega)n^{3/2}}{E-\tilde{V}(x)}} \tag{6.87}$$

$$\times\left[c\left(12\sqrt{n}+1\right)\frac{r_{\tilde{V},E}(x)}{E-\tilde{V}(x)}\left|\frac{\partial \phi}{\partial w^i}(w)\right|s + 2\sqrt{n}\int_0^s \left|\frac{\partial Q}{\partial w^i}(\sigma, x, w)\right| d\sigma\right],$$

for $(s, x, w) \in [0, t_0+\varepsilon[\times B(x_0, \varepsilon) \times U$.

From (6.84), it follows that $Q(s, x, w) = x + \int_0^s \bar{Q}(\sigma, x, w)d\sigma$, for $(s, x, w) \in [0, t_0+\varepsilon[\times B(x_0, \varepsilon) \times U$. Hence $\frac{\partial Q}{\partial w^i}(s, x, w) = \int_0^s \frac{\partial \bar{Q}}{\partial w^i}(\sigma, x, w)d\sigma$, for $(s, x, w) \in [0, t_0+\varepsilon[\times B(x_0, \varepsilon) \times U$. This latter equality and (6.87) imply

$$\left|\frac{\partial Q}{\partial w^i}(s, x, w) - \frac{r_{\tilde{V},E}(x)}{\left(\frac{E-\tilde{V}(x)}{c^2}\right)}\frac{\partial \phi}{\partial w^i}(w)s\right| \leq \frac{2c^2n^{3/2}\beta(\tilde{V}, \tilde{B}, \Omega)}{E-\tilde{V}(x)} \tag{6.88}$$

$$e^{\frac{10\delta(\Omega)\beta(\tilde{V}, \tilde{B}, \Omega)n^{3/2}}{E-\tilde{V}(x)}}\left[c\left(12\sqrt{n}+1\right)\frac{r_{\tilde{V},E}(x)}{E-\tilde{V}(x)}\left|\frac{\partial \phi}{\partial w^i}(w)\right|\frac{s^2}{2}\right.$$

$$\left.+2\sqrt{n}\int_0^s\int_0^\tau \left|\frac{\partial Q}{\partial w^i}(\sigma, x, w)\right| d\sigma d\tau\right],$$

for $(s, x, w) \in [0, t_0+\varepsilon[\times B(x_0, \varepsilon) \times U$.

Note that

$$\int_0^s \int_0^\tau \left| \frac{\partial Q}{\partial w^i}(\sigma, x, w) \right| d\sigma d\tau \leq s \int_0^s \left| \frac{\partial Q}{\partial w^i}(\sigma, x, w) \right| d\sigma \tag{6.89}$$
$$\leq \frac{5\delta(\Omega)}{c} \int_0^s \left| \frac{\partial Q}{\partial w^i}(\sigma, x, w) - \sigma \frac{r_{\tilde{V},E}(x)}{\left(\frac{E-\tilde{V}(x)}{c^2}\right)} \frac{\partial \phi}{\partial w^i}(w) \right| d\sigma$$
$$+ \frac{5r_{\tilde{V},E}(x)\delta(\Omega)}{c\left(\frac{E-\tilde{V}(x)}{c^2}\right)} \left| \frac{\partial \phi}{\partial w^i}(w) \right| \frac{s^2}{2},$$

for $(s, x, w) \in [0, t_0 + \varepsilon[\times B(x_0, \varepsilon) \times U$ (we used that $t_0 + \varepsilon \leq \frac{5\delta(\Omega)}{c}$).

Let

$$C_3'(E, x, \tilde{V}, \tilde{B}, \Omega) = 20cn^2\beta(\tilde{V}, \tilde{B}, \Omega)\delta(\Omega)e^{\frac{10\delta(\Omega)\beta(\tilde{V},\tilde{B},\Omega)n^{3/2}}{E-\tilde{V}(x)}} \tag{6.90}$$

$$C_4'(E, x, \tilde{V}, \tilde{B}, \Omega) = 2c^2n^{3/2}\beta(\tilde{V}, \tilde{B}, \Omega)e^{\frac{10\delta(\Omega)\beta(\tilde{V},\tilde{B},\Omega)n^{3/2}}{E-\tilde{V}(x)}} \tag{6.91}$$
$$\times \left[c\left(12\sqrt{n} + 1\right) \frac{r_{\tilde{V},E}(x)}{E - \tilde{V}(x)} + \frac{10r_{\tilde{V},E}(x)\sqrt{n}\delta(\Omega)}{c\left(\frac{E-\tilde{V}(x)}{c^2}\right)} \right],$$
$$C_5'(E, x, \tilde{V}, \tilde{B}, \Omega) = C_4'(E, x, \tilde{V}, \tilde{B}, \Omega)e^{\frac{C_3'(E,x,\tilde{V},\tilde{B},\Omega)}{E-\tilde{V}(x)}}, \tag{6.92}$$

for $x \in \Omega$.

From (6.88)-(6.92) and Gronwall's lemma (formulated in Subsection 6.7), it follows that

$$\left| \frac{\partial Q}{\partial w^i}(s, x, w) - \frac{r_{\tilde{V},E}(x)}{\left(\frac{E-\tilde{V}(x)}{c^2}\right)} \frac{\partial \phi}{\partial w^i}(w)s \right| \leq \frac{C_5'(E, x, \tilde{V}, \tilde{B}, \Omega)}{E - \tilde{V}(x)} \left| \frac{\partial \phi}{\partial w^i}(w) \right| \frac{s^2}{2} \tag{6.93}$$

for $(s, x, w) \in [0, t_0 + \varepsilon[\times B(x_0, \varepsilon) \times U$.

Now we assume without loss of generality that the integer k in (6.76) is n, and $p_0^n > 0$. We remind that $w_0 \in U$ is defined by $\phi(w_0) = \frac{p_0}{|p_0|}$. We shall prove (6.99).

From (6.77), it follows that

$$\frac{\partial \phi}{\partial w^l}(w_0) = e_l - \frac{w_0^l}{\sqrt{1 - |w_0|^2}} e_n, \tag{6.94}$$
$$|\frac{\partial \phi}{\partial w^l}(w_0)| \leq \sqrt{1 + n}, \tag{6.95}$$

for $l = 1 \dots n-1$ and where $(e_1, \dots, e_n)$ is the canonical basis of $\mathbb{R}^n$ and $w_0 = (w_0^1, \dots, w_0^{n-1})$ (for (6.95), we used the estimate $\frac{p_0^n}{|p_0|} \geq n^{-1/2}$ which implies that $1 - |w_0|^2 \geq \frac{1}{n}$ and we used $|w_0^l| \leq |w_0| < 1$, $l = 1 \dots n-1$ and we used (6.94)). In addition, using (6.94), we obtain

$$|\sum_{l=1}^{n-1} \mu_l \frac{\partial \phi}{\partial w^l}(w_0)| \geq |(\mu_1, \dots, \mu_{n-1})|, \tag{6.96}$$

for all $(\mu_1, \dots, \mu_{n-1}) \in \mathbb{R}^{n-1}$.

Using the fact that $\phi(w_0) \in \mathbb{S}^{n-1}$ is orthogonal to $\frac{\partial \phi}{\partial w^l}(w_0)$, $l = 1 \dots n-1$, and using (6.96), we obtain

$$|\mu_1 \phi(w_0) + \sum_{l=1}^{n-1} \mu_{l+1} \frac{\partial \phi}{\partial w^l}(w_0)| = \sqrt{\mu_1^2 + |\sum_{l=1}^{n-1} \mu_{l+1} \frac{\partial \phi}{\partial w^l}(w_0)|^2} \geq n^{-1/2} \sum_{l=1}^{n} |\mu_l|, \tag{6.97}$$

for all $(\mu_1, \dots, \mu_n) \in \mathbb{R}^n$.

Note that

$$\begin{aligned} &\left| \lambda_1 \frac{\partial Q}{\partial t}(t_0, x_0, w_0) + \sum_{l=1}^{n-1} \lambda_{l+1} \frac{\partial Q}{\partial w^l}(t_0, x_0, w_0) \right| \geq \qquad (6.98) \\ &\left| \lambda_1 g(r_{\tilde{V},E}(x_0)\phi(w_0)) + \sum_{l=1}^{n-1} \lambda_{l+1} \frac{c^2 r_{\tilde{V},E}(x_0) t_0}{E - \tilde{V}(x_0)} \frac{\partial \phi}{\partial w^l}(t_0, x_0, w_0) \right| \\ &-|\lambda_1| \left| \frac{\partial Q}{\partial t}(t_0, x_0, w_0) - g(r_{\tilde{V},E}(x_0)\phi(w_0)) \right| \\ &-\sum_{l=1}^{n-1} |\lambda_{l+1}| \left| \frac{\partial Q}{\partial w^l}(t_0, x_0, w_0) - \frac{c^2 r_{\tilde{V},E}(x_0) t_0}{E - \tilde{V}(x_0)} \frac{\partial \phi}{\partial w^l}(t_0, x_0, w_0) \right|, \end{aligned}$$

for $(\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n$.

We estimate the first term on the right-hand side of (6.98) by using (6.97) (note that $g(r_{\tilde{V},E}(x_0)\phi(w_0)) = \frac{c^2 r_{\tilde{V},E}(x_0)}{E - \tilde{V}(x_0)} \phi(w_0)$). We estimate the second term and third term on the right-hand side of (6.98) by using (6.82) and (6.93) and (6.95). Using also the estimate $t_0 \leq \frac{5\delta(\Omega)}{c}$, we finally obtain

$$\left| \lambda_1 \frac{\partial Q}{\partial t}(t_0, x_0, w_0) + \sum_{l=1}^{n-1} \lambda_{l+1} \frac{\partial Q}{\partial w^l}(t_0, x_0, w_0) \right| \geq |\lambda_1| \frac{c C_6}{\sqrt{n}} + t_0 \sum_{l=1}^{n-1} |\lambda_{l+1}| \frac{C_7}{\sqrt{n}}, \tag{6.99}$$

for $(\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n$. This latter inequality and (6.26) and $t_0 > 0$ imply that the family $\left(\frac{\partial Q}{\partial t}(t_0, x_0, w_0), \frac{\partial Q}{\partial w^1}(t_0, x_0, w_0), \dots, \frac{\partial Q}{\partial w^{n-1}}(t_0, x_0, w_0)\right)$ is free. Then using inverse function theorem, it follows that φ_E is a local C^1 diffeomorphism at (t_0, x_0, p_0).

Proposition 6.3 is proved. $\square$

6.9 *Proof of Proposition 6.4.* Before proving Proposition 6.4, we shall first prove the following Lemma 6.1.

Lemma 6.1. *Assume that*

$$\begin{aligned} &E \geq C_1(\tilde{V}, \tilde{B}, \Omega), \\ &C_8(E, \tilde{V}, \tilde{B}, \Omega) > 0, \end{aligned} \tag{6.100}$$

where C_1 and C_8 are defined by (6.8) *and* (6.15)*.*

Let $x \in \Omega$ and $p \in \mathbb{S}^{n-1}_{x,E}$. Then

$$\psi_2(t_{+,x,p}, x, p) \circ N(\psi_1(t_{+,x,p}, x, p)) > 0, \tag{6.101}$$

where $t_{+,x,p}$ is defined by (6.21) *and $\psi = (\psi_1, \psi_2)$ is the flow of the differential system* (5.1)*, and where $N(y)$ denotes the unit outward normal vector of ∂D at $y \in \partial D$ ($\circ$ denotes the usual scalar product on $\mathbb{R}^n$).*

Proof of Lemma 6.1. Consider the function $m \in C^2(\mathbb{R}, \mathbb{R})$ defined by

$$m(t) = \chi_\Omega(\psi_1(t, x, p)),\ \ t \in \mathbb{R}, \tag{6.102}$$

where χ_Ω is a C^2 defining function for Ω (see definition of C_8, (6.15)). Derivating twice (6.102) and using (5.1), we obtain

$$\begin{aligned} &\ddot{m}(t) = Hess\chi_\Omega(\psi_1(t, x, p))(g(\psi_2(t, x, p)), g(\psi_2(t, x, p))) \qquad (6.103)\\ &+ \left(1 + \frac{|\psi_2(t, x, p)|^2}{c^2}\right)^{-1/2} \nabla\chi_\Omega(\psi_1(t, x, p)) \circ F\,(\psi_1(t, x, p),\\ &g(\psi_2(t, x, p))) - \frac{\psi_2(t, x, p) \circ F\,(\psi_1(t, x, p), g(\psi_2(t, x, p)))}{c^2\left(1 + \frac{|\psi_2(t,x,p)|^2}{c^2}\right)^{3/2}}\\ &\times\nabla\chi_\Omega(\psi_1(t, x, p)) \circ \psi_2(t, x, p), \end{aligned}$$

for $t \in \mathbb{R}$ and where g is the function defined by (5.6). From (6.103), conservation of energy and (6.6) and $|g(\psi_2(t, x, p))| < c$, it follows that $\ddot{m}(t_{+,x,p}) \geq c^2 C_8(E, \tilde{V}, \tilde{B}, \Omega) > 0$ (we used (6.100)).

For all $t \in [0, t_{+,x,p}[$, $\psi_1(t, x, p) \in \Omega$. Hence $m(t) < 0$, $t \in [0, t_{+,x,p}[$. This estimate and the estimate $\ddot{m}(t_{+,x,p}) > 0$ and Taylor expansion of m at $t_{+,x,p}$ ($m(t) = \dot{m}(t_{+,x,p})(t - t_{+,x,p}) + \frac{1}{2}\ddot{m}(t_{+,x,p})(t - t_{+,x,p})^2 + o((t - t_{+,x,p})^2),\ t \in \mathbb{R}$) imply that $\dot{m}(t_{+,x,p}) > 0$.

Lemma 6.1 is proved. $\square$

Now we are ready to prove Proposition 6.4.

Let $x \in \Omega$. As $n \geq 2$, the set $\Omega \backslash \{x\}$ is connected and we shall prove that the set

$$\begin{aligned} A_x &= \{y \in \Omega\backslash\{x\} \mid \text{there exists } p \in \mathbb{S}^{n-1}_{x,E} \text{ and } t > 0 \text{ such that } (t,x,p) \in \Lambda \\ &\quad \text{and } \psi_1(t,x,p) = y\} \end{aligned}$$

is a closed and open nonempty subset of $\Omega\backslash\{x\}$ (where $\psi = (\psi_1, \psi_2)$ is the differential flow of (5.1)). Then we will have $A_x = \Omega\backslash\{x\}$, which will prove Proposition 6.4.

Note that A_x is nonempty since for $p \in \mathbb{S}^{n-1}_{x,E}$, $(0,x,p) \in \Lambda$ and $\frac{\partial \psi_1}{\partial t}(t,x,p)_{|t=0} = g(p) \neq 0$. Hence there exists $\varepsilon > 0$ such that $(\varepsilon, x, p) \in \Lambda$ and $\psi_1(\varepsilon, x, p) \neq \psi_1(0,x,p) = x$.

Note also that A_x is an open subset of $\Omega\backslash\{x\}$. Let $y \in A_x$. Then there exists $p \in \mathbb{S}^{n-1}_{x,E}$ and $t > 0$ such that $(t,x,p) \in \Lambda$ and $\psi_1(t,x,p) = y$. From (6.27) and Proposition 6.3, it follows, in particular, that there exists an open neighborhood $U \subseteq \Omega\backslash\{x\}$ of y such that $U \subseteq A_x$.

It remains to prove that A_x is a closed subset of $\Omega\backslash\{x\}$. Consider a sequence (y_k) of points of $\Omega\backslash\{x\}$ which converges to some $y \in \Omega\backslash\{x\}$ as $k \to +\infty$. For each k, there exists $p_k \in \mathbb{S}^{n-1}_{x,E}$ and $t_k > 0$ such that $(t_k, x, p_k) \in \Lambda$ and

$$\psi_1(t_k, x, p_k) = y_k. \tag{6.104}$$

From Proposition 6.1, it follows that $t_k \in [0, \frac{5\delta(\Omega)}{c}]$ for all k. Using compactness of $[0, \frac{5\delta(\Omega)}{c}]$ and compactness of $\mathbb{S}^{n-1}_{x,E}$, we can assume that (t_k) converges to some $t \in [0, \frac{5\delta(\Omega)}{c}]$ and that (p_k) converges to some $p \in \mathbb{S}^{n-1}_{x,E}$. Using (6.104) and continuity of ψ_1, we obtain

$$y = \lim_{k\to+\infty} y_k = \lim_{k\to+\infty} \psi_1(t_k, x, p_k) = \psi_1(t,x,p). \tag{6.105}$$

Note that $t > 0$ since $y \neq x$. Let $s \in [0, t[$. Then using that $t_k \to t$ as $k \to +\infty$, we obtain that there exists a rank N_s such that $s < t_k$ for $k \geq N_s$. Hence $(s, x, p_k) \in \Lambda$ for $k \geq N_s$ and, in particular, $\psi_1(s, x, p_k) \in \Omega$ for $k \geq N_s$. Hence we obtain that

$$\psi_1(s,x,p) = \lim_{k\to+\infty} \psi_1(s,x,p_k) \in \bar{\Omega}, \text{ for } s \in [0,t[. \tag{6.106}$$

Using Lemma 6.1 with (6.105) ($y \in \Omega$) and (6.106), we obtain that $(t,x,p) \in \Lambda \cap \left(]0,+\infty[\times\{x\} \times \mathbb{S}^{n-1}_{x,E}\right)$ and $\psi_1(t,x,p) = y$. Hence $y \in A_x$.

Proposition 6.4 is proved. $\square$

7 The nonrelativistic case

7.1 *Nonrelativistic Newton equation in electromagnetic field.* Consider the classical nonrelativistic Newton equation in a static electromagnetic field in an open subset Ω of $\mathbb{R}^n$, $n \geq 2$,

$$\ddot{x} = -\nabla V(x) + B(x)\dot{x}, \tag{7.1}$$

where $x = x(t)$ is a C^1 function with values in Ω, $\dot{x} = \frac{dx}{dt}$, and $V \in C^2(\bar{\Omega}, \mathbb{R})$, $B \in \mathcal{F}_{mag}(\bar{\Omega})$.

The equation (7.1) is an equation for $x = x(t)$ and is the equation of motion in $\mathbb{R}^n$ of a nonrelativistic particle of mass $m = 1$ and charge $e = 1$ in an external electromagnetic field described by V and B. In this equation x is the position of the particle, $\dot{x}$ is its velocity, t is the time.

For the equation (7.1) the energy

$$E = \frac{1}{2}|\dot{x}(t)|^2 + V(x(t)) \tag{7.2}$$

is an integral of motion.

7.2 *Inverse boundary problem.* Consider equation (7.1) under condition (1.3).

One can prove that at sufficiently large energy E (i.e. $E > E^{nr}(\|V\|_{C^2,D}, \|B\|_{C^1,D}, D) \geq \sup_{x\in D} V(x)$ where real constant $E^{nr}(\|V\|_{C^2,D}, \|B\|_{C^1,D}, D)$ also has properties (2.3) and (1.7)), the solutions x of energy E have properties (2.1) and (2.2) (the proof is obtained by slight modifications of proofs of Section 6). Then at fixed energy $E > E^{nr}(\|V\|_{C^2,D}, \|B\|_{C^1,D}, D)$, one can define $s^{nr}_{V,B}(E, q_0, q)$, $k^{nr}_{0,V,B}(E, q_0, q)$, $k^{nr}_{V,B}(E, q_0, q)$, as were defined $s_{V,B}(E, q_0, q)$, $k_{0,V,B}(E, q_0, q)$, $k_{V,B}(E, q_0, q)$, in Section 2 for any $q_0, q \in \bar{D}$, $q_0 \neq q$. Further one can consider the following nonrelativistic version of Problem 1 formulated in Introduction

Problem $1'$: given $k^{nr}_{V,B}(E, q_0, q)$, $k^{nr}_{0,V,B}(E, q_0, q)$ for all $q_0, q \in \partial D$, $q_0 \neq q$, at fixed sufficiently large energy E, find V and B.

The following uniqueness theorem holds

Theorem 7.1. *At fixed $E > E^{nr}(\|V\|_{C^2,D}, \|B\|_{C^1,D}, D)$, the boundary data $k^{nr}_{V,B}(E, q_0, q)$, $(q_0, q) \in \partial D \times \partial D$, $q_0 \neq q$, uniquely determine V, B.*

At fixed $E > E^{nr}(\|V\|_{C^2,D}, \|B\|_{C^1,D}, D)$, the boundary data $k^{nr}_{0,V,B}(E, q_0, q)$, $(q_0, q) \in \partial D \times \partial D$, $q_0 \neq q$, uniquely determine V, B.

Theorem 7.1 is proved in Subsection 7.6.

7.3 *Inverse scattering problem.* We consider equation (7.1) under condition (1.4).

The following is valid (see, for example, [LT] where classical scattering of particles in a long-range magnetic field is studied, and see [S] where classical scattering of particles in a short-range electric field is studied): for any $(v_-,x_-)\in\mathbb{R}^n\times\mathbb{R}^n,\ v_-\neq 0$, the equation (7.1) has a unique solution $x\in C^2(\mathbb{R},\mathbb{R}^n)$ such that

$$x(t)=v_-t+x_-+y_-(t), \tag{7.3}$$

where $\dot y_-(t)\to 0,\ y_-(t)\to 0,$ as $t\to-\infty$; in addition for almost any $(v_-,x_-)\in\mathbb{R}^n\times\mathbb{R}^n,\ v_-\neq 0,$

$$x(t)=v_+t+x_++y_+(t), \tag{7.4}$$

where $v_+\neq 0,\ v_+=a^{nr}(v_-,x_-),\ x_+=b^{nr}(v_-,x_-),\ \dot y_+(t)\to 0,\ \ y_+(t)\to 0,$ as $t\to+\infty$.

For an energy $E>0$, the map $S_E^{nr}:\mathbb{S}_E^{nr}\times\mathbb{R}^n\to\mathbb{S}_E^{nr}\times\mathbb{R}^n$ (where $\mathbb{S}_E^{nr}=\{v\in\mathbb{R}^n\mid |v|=\sqrt{2E}\}$) given by the formulas

$$v_+=a^{nr}(v_-,x_-),\ x_+=b^{nr}(v_-,x_-), \tag{7.5}$$

is called the scattering map at fixed energy E for the equation (7.1) under condition (1.4). By $\mathcal{D}(S_E^{nr})$ we denote the domain of definition of S_E^{nr}. The data $a^{nr}(v_-,x_-)$, $b^{nr}(v_-,x_-)$ for $(v_-,x_-)\in\mathcal{D}(S_E^{nr})$ are called the scattering data at fixed energy E for the equation (7.1) under condition (1.4). We consider the following inverse scattering problem at fixed energy for the equation (7.1) under condition (1.4):

$$\textbf{Problem } 2':\ \text{given } S_E^{nr} \text{ at fixed energy } E,\ \text{find } V \text{ and } B.$$

From Theorem 7.1 and property (1.7), we obtain

Theorem 7.2. *Let $\lambda\in\mathbb{R}^+$ and let D be a bounded strictly convex (in the strong sense) open domain of $\mathbb{R}^n$ with C^2 boundary. Let $V_1,V_2\in C_0^2(\mathbb{R}^n,\mathbb{R})$, $B_1,B_2\in C_0^1(\mathbb{R}^n,A_n(\mathbb{R}))\cap\mathcal{F}_{mag}(\mathbb{R}^n)$ $\max(\|V_1\|_{C^2,D},\|V_2\|_{C^2,D},\|B_1\|_{C^1,D},\|B_2\|_{C^1,D})\leq\lambda$, and $\mathrm{supp}(V_1)\cup\mathrm{supp}(V_2)\cup\mathrm{supp}(B_1)\cup\mathrm{supp}(B_2)\subseteq D$. Let S_E^μ be the (nonrelativistic) scattering map at fixed energy E subordinate to (V_μ,B_μ) for $\mu=1,2$. Then there exists a nonnegative real constant $E^{nr}(\lambda,D)$ such that for any $E>E^{nr}(\lambda,D)$, $(V_1,B_1)\equiv(V_2,B_2)$ if and only if $S_E^1\equiv S_E^2$.*

7.4 *Classical Hamiltonian mechanics.* For $x\in\bar D$ and for $E>V(x)$, we define

$$r_{V,E}^{nr}(x)=\sqrt{2(E-V(x))}.$$

Let $\mathbf{A}\in\mathcal{F}_{pot}(D,B)$. The equation (7.1) in D is the Euler-Lagrange equation for the Lagrangian L^{nr} defined by $L^{nr}(\dot x,x)=\frac{1}{2}|\dot x|^2+\mathbf{A}(x)\circ\dot x-V(x)$, $\dot x\in\mathbb{R}^n$

and $x \in D$, where $\circ$ denotes the usual scalar product on $\mathbb{R}^n$. The Hamiltonian H^{nr} associated to the Lagrangian L^{nr} by Legendre's transform (with respect to $\dot x$) is $H^{nr}(P,x) = \frac{1}{2}|P - \mathbf{A}(x)|^2 + V(x)$ where $P \in \mathbb{R}^n$ and $x \in D$. Then equation (1.1) in D is equivalent to the Hamilton's equation

$$\begin{aligned} \dot x &= \frac{\partial H^{nr}}{\partial P}(P,x),\\ \dot P &= -\frac{\partial H^{nr}}{\partial x}(P,x), \end{aligned} \tag{7.6}$$

for $P \in \mathbb{R}^n$, $x \in D$.

For a solution $x(t)$ of equation (7.1) in D, we define the impulse vector

$$P^{nr}(t) = \dot x(t) + \mathbf{A}(x(t)).$$

Further for $q_0, q \in \bar D$, $q_0 \neq q$, and $t \in [0, s^{nr}(E, q_0, q)]$, we consider

$$P^{nr}(t,E,q_0,q) = \dot x^{nr}(t,E,q_0,q) + \mathbf{A}(x^{nr}(t,E,q_0,q)), \tag{7.7}$$

where $x^{nr}(.,E,q_0,q)$ is the solution given by (2.2) in the nonrelativistic case. From Maupertuis's principle (see [A]), it follows that if $x(t)$, $t \in [t_1,t_2]$, is a solution of (7.1) in D with energy E, then $x(t)$ is a critical point of the functional $\mathcal{A}(y) = \int_{t_1}^{t_2} \left[r^{nr}_{V,E}(y(t))|\dot y(t)| + \ \mathbf{A}(y(t)) \circ \dot y(t)\right] dt$ defined on the set of the functions $y \in C^1([t_1,t_2], D)$, with boundary conditions $y(t_1) = x(t_1)$ and $y(t_2) = x(t_2)$. Note that for $q_0, q \in D$, $q_0 \neq q$, functional $\mathcal{A}$ taken along the trajectory of the solution $x^{nr}(.,E,q_0,q)$ given by (2.2) is equal to the reduced action $\mathcal{S}^{nr}_{0_{V,\mathbf{A},E}}(q_0,q)$ from q_0 to q at fixed energy E for (7.6), where

$$\mathcal{S}^{nr}_{0_{V,\mathbf{A},E}}(q_0,q) = \begin{cases} 0, & \text{if } q_0 = q,\\ \int_0^{s^{nr}(E,q_0,q)} P^{nr}(s,E,q_0,q) \circ \dot x^{nr}(s,E,q_0,q) ds, & \text{if } q_0 \neq q, \end{cases} \tag{7.8}$$

for $q_0, q \in \bar D$.

7.5 *Properties of the reduced action at a fixed and sufficiently large energy.* The reduced action at fixed and sufficiently large energy for (7.6) has the same properties that those given in Proposition 3.1, 3.2 for the reduced action at fixed and sufficiently large energy for the relativistic case.

Let $E > E^{nr}(\|V\|_{C^2,D}, \|B\|_{C^1,D}, D)$. The reduced action $\mathcal{S}^{nr}_{0_{V,\mathbf{A},E}}$ at fixed energy E has the following properties:

$$\mathcal{S}^{nr}_{0_{V,\mathbf{A},E}} \in C(\bar{D}\times\bar{D},\mathbb{R}), \tag{7.9}$$
$$\mathcal{S}^{nr}_{0_{V,\mathbf{A},E}} \in C^2((\bar{D}\times\bar{D})\backslash\bar{G},\mathbb{R}), \tag{7.10}$$
$$\frac{\partial \mathcal{S}^{nr}_{0_{V,\mathbf{A},E}}}{\partial x_i}(\zeta,x) = k^{nr,i}_{V,B}(E,\zeta,x) + \mathbf{A}_i(x), \tag{7.11}$$
$$\frac{\partial \mathcal{S}^{nr}_{0_{V,\mathbf{A},E}}}{\partial \zeta_i}(\zeta,x) = -k^{nr,i}_{0,V,B}(E,\zeta,x) - \mathbf{A}_i(\zeta), \tag{7.12}$$
$$\frac{\partial^2 \mathcal{S}^{nr}_{0_{V,\mathbf{A},E}}}{\partial \zeta_i\partial x_j}(\zeta,x) = -\frac{\partial k^{nr,i}_{0,V,B}}{\partial x_j}(E,\zeta,x) = \frac{\partial k^{nr,j}_{V,B}}{\partial \zeta_i}(E,\zeta,x), \tag{7.13}$$

for $(\zeta,x)\in(\bar{D}\times\bar{D})\backslash\bar{G}$, $\zeta=(\zeta_1,..,\zeta_n)$, $x=(x_1,..,x_n)$, and $i,j=1\ldots n$. In addition,

$$\max\left(\left|\frac{\partial \mathcal{S}^{nr}_{0_{V,\mathbf{A},E}}}{\partial x_i}(\zeta,x)\right|,\left|\frac{\partial \mathcal{S}^{nr}_{0_{V,\mathbf{A},E}}}{\partial \zeta_i}(\zeta,x)\right|\right) \le M_1, \tag{7.14}$$
$$\left|\frac{\partial^2 \mathcal{S}^{nr}_{0_{V,\mathbf{A},E}}}{\partial \zeta_i\partial x_j}(\zeta,x)\right| \le \frac{M_2}{|\zeta-x|}, \tag{7.15}$$

for $(\zeta,x)\in(\bar{D}\times\bar{D})\backslash\bar{G}$, $\zeta=(\zeta_1,..,\zeta_n)$, $x=(x_1,..,x_n)$, and $i,j=1\ldots n$, and where M_1 and M_2 depend on V, B and D.

In addition, the map $\nu_{V,B,E}:\partial D\times D\to\mathbb{S}^{n-1}$, defined by

$$\nu^{nr}_{V,B,E}(\zeta,x) = -\frac{k^{nr}_{V,B}(E,\zeta,x)}{|k^{nr}_{V,B}(E,\zeta,x)|},\ \text{for } (\zeta,x)\in\partial D\times D, \tag{7.16}$$

has the following properties:

$$\begin{aligned}&\nu^{nr}_{V,B,E}\in C^1(\partial D\times D,\mathbb{S}^{n-1}),\\ &\text{the map } \nu^{nr}_{V,B,E,x}:\partial D\to\mathbb{S}^{n-1},\ \zeta\mapsto\nu^{nr}_{V,B,E}(\zeta,x),\ \text{is a}\\ &C^1 \text{ orientation preserving diffeomorphism from } \partial D \text{ onto } \mathbb{S}^{n-1}\end{aligned} \tag{7.17}$$

for $x\in D$ (where we choose the canonical orientation of $\mathbb{S}^{n-1}$ and the orientation of ∂D given by the canonical orientation of $\mathbb{R}^n$ and the unit outward normal vector).

Remark 7.1. Equalities (7.11) and (7.12) are known formulas of classical Hamiltonian mechanics (see Section 46 and further Sections of [A]).

Remark 7.2. Taking account of (7.11) and (7.12), we obtain the following formulas: at $E > E^{nr}(\|V\|_{C^2,D}, \|B\|_{C^1,D}, D)$, for any $x,\ \zeta \in \bar{D}$, $x \neq \zeta$,

$$\begin{aligned}
B_{i,j}(x) &= -\frac{\partial k^{nr,j}_{V,B}}{\partial x_i}(E,\zeta,x) + \frac{\partial k^{nr,i}_{V,B}}{\partial x_j}(E,\zeta,x),\\
B_{i,j}(x) &= -\frac{\partial k^{nr,j}_{0,V,B}}{\partial x_i}(E,x,\zeta) + \frac{\partial k^{nr,i}_{0,V,B}}{\partial x_j}(E,x,\zeta).
\end{aligned}$$

7.6 *Proof of Theorem 7.1.* We define the $n-1$ differential form $\omega^{nr}_{0,V,B}$ on $\partial D \times D$ as was defined the $n-1$ differential form $\omega_{0,V,B}$ on $\partial D \times D$ in Subsection 3.3.

Now let $\lambda \in \mathbb{R}^+$ and $V_1, V_2 \in C^2(\bar{D}, \mathbb{R})$, $B_1, B_2 \in \mathcal{F}_{mag}(\bar{D})$ such that $\max(\|V_1\|_{C^2,D}, \|V_2\|_{C^2,D}, \|B_1\|_{C^1,D}, \|B_2\|_{C^1,D}) \leq \lambda$. For $\mu = 1,2$, let $\mathbf{A}_\mu \in \mathcal{F}_{pot}(D, B_\mu)$.

Let $E > E^{nr}(\lambda, \lambda, D)$ where $E^{nr}(\lambda, \lambda, D)$ is defined by the nonrelativistic formulation of (1.7). Consider $\beta^{nr,1}$, $\beta^{nr,2}$ the differential one forms defined on $(\partial D \times \bar{D})\backslash\bar{G}$ by

$$\beta^{nr,\mu}(\zeta, x) = \sum_{j=1}^{n} k^{nr,j}_{V_\mu,B_\mu}(E,\zeta,x)dx_j, \tag{7.18}$$

for $(\zeta, x) \in (\partial D \times \bar{D})\backslash\bar{G}$, $x = (x_1, \ldots, x_n)$ and $\mu = 1,2$.

Define the differential forms Φ^{nr}_0 and Φ^{nr} as were defined Φ_0 and Φ in Subsection 3.3 (replace β^μ, $\mathcal{S}_{0_{V_\mu,\mathbf{A}_\mu,E}}$ by $\beta^{nr,\mu}$, $\mathcal{S}^{nr}_{0_{V_\mu,\mathbf{A}_\mu,E}}$, $\mu = 1,2$).

Then Lemma 3.1 and Theorem 3.1 remain valid by replacing Φ_0, Φ, $r_{V_\mu,E}$, ω_{0,V_μ,B_μ} and $\bar{k}_{V_\mu,B_\mu}$ by Φ^{nr}_0, Φ^{nr}, $r^{nr}_{V_\mu,E}$, $\omega^{nr}_{0,V_\mu,B_\mu}$ and $k^{nr}_{V_\mu,B_\mu}$, and the proof of these results are obtained by slight modifications of proof of Lemma 3.1 and Theorem 3.1.

Hence Theorem 7.1 follows from the nonrelativistic formulation of Theorem 3.1. $\square$

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